AIRPLANE STRUCTURES

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To

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HAYWARD PROFESSOR OF CIVIL ENGINEERING,
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AS AN EXPRESSION OF APPRECIATION
FOR THE TRAINING IN THE THEORY OF STRUCTURES
GIVEN TO THE AUTHORS WHILE STUDENTS IN HIS COURSE

PREFACE

This volume has been prepared with a twofold purpose. In order to meet the needs of instructors giving courses in airplane structures it emphasizes and describes the application of the fundamental principles of structural theory to practical airplane design. At the same time considerable data on allowable stresses in various materials and allowable loads on standardized airplane parts are included so that it may satisfy the ordinary requirements of the practicing engineer. In other words, an attempt has been made to combine an exposition of basic structural theory with practical design information sufficient to solve the more common problems of the aeronautical structural engineer.

As a text-book, this volume is intended for use by students who are well grounded in the Mechanics of Materials and Calculus. While such men should understand the computation of reactions and the construction of curves of shear and bending moment, experience has shown that a review of these subjects is not only desirable but necessary. Although of somewhat limited application in the field of aeronautical structural analysis, the use of Influence Lines has been found to be of great assistance in clarifying a student's conception of shear and bending moment, and methods for constructing such lines have been included for the benefits to be derived from their study.

Methods of analysis for both statically determinate and statically indeterminate frameworks are treated in the text with sufficient numerical examples to illustrate the basic methods of computation for the analysis of conventional structures as well as the more important deviations from the conventional. These examples do not cover each of the parts of an airplane as it is assumed that the student will recognize the method of analysis illustrated for a nacelle framework to be equally applicable to a fuselage structure, etc. Space which might have been devoted to a sample stress analysis of a complete airplane has therefore been utilized to give a more thorough exposition of fundamental theory. Emphasis on detailed methods of computation has been avoided, but sufficient specific applications have been described to cover most of the types of construction in common use. In several cases suggestions have been made as to methods of tabulating work to reduce to a minimum the labor of arithmetical computations.

The discussion of the external loads to which an airplane is subjected has been confined to a brief analysis of the critical loading conditions.

It was considered unwise to insert specific rules for obtaining these loads since they are continually being changed in detail. Those applicable to commercial types at any date may be obtained directly from the Aeronautics Branch of the Department of Commerce and for military types from the Bureau of Aeronautics of the Navy Department at Washington, D. C., or the Materiel Division of the Army Air Corps at Dayton, Ohio.

In many ways this volume is a revised edition of "Airplane Design" published by the Air Corps. The chapter on performance estimation in that book has been omitted entirely and that on external loads greatly condensed. The more useful tables and charts of structural design data, however, have been retained, and several new ones have been added. The chapters on structural theory have been greatly expanded and their scope extended.

Suggestive problems which are, in a majority of cases, based on the illustrative examples worked out in the text have been placed at the end of each chapter. They will be found to help the student make a more careful study of the examples in the text by carrying out parallel computations with slightly different data. They will also be of benefit to the "home student" who wishes to determine whether or not he understands the principles involved.

The authors are greatly indebted to the Air Corps, U. S. Army, for permission to use much material from their publications and for the loan of cuts for many of the figures taken from "Airplane Design." An attempt has been made to acknowledge the sources for all charts and tables of structural design data not taken directly from "Airplane Design," and where no acknowledgment is given the data may be assumed to have been taken from that source. Thanks are also due to many organizations and individuals both in the Government services and out of it for help and contributions of material.

The authors are particularly grateful to the manufacturers who furnished detailed weight data on their products so that the chapter on Weight Estimation might be adequate. The manufacturers of equipment are mentioned in connection with the data they furnished. Because of the method used for classifying the airplanes studied, and in view of the confidential nature of the data, it has been thought best to make no specific acknowledgments to the manufacturers who furnished detailed weights of their airplanes, although the authors fully appreciate the debt which they owe for such assistance.

Alfred S. Niles,
Joseph S. Newell.

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AIRPLANE STRUCTURES

CHAPTER I

GENERAL DESIGN PROCEDURE

The purpose of this chapter is to give in brief outline the steps gone through in the design of a new airplane. No exhaustive treatment will be given any of these steps, the object being more to show the design procedure as a whole and the interrelations of its various phases. Those phases which are most intimately connected with the structural design, as distinct from the aerodynamic design, of the airplane and the physical principles and methods of computation used in the structural design will be explained in the later chapters of this volume. For the aerodynamic design the reader is referred to the text-books on that subject, such as Montieth, "Simple Aerodynamics and the Airplane"; Bairstow, "Applied Aerodynamics"; Diehl, "Engineering Aerodynamics"; Warner, "Airplane Design"; etc.

1 : 1. Types of Design Procedure — A new airplane design may be entirely new or it may be a revision of an existing design. The chief difference in the procedure in the two cases is that in a revision, a large part of the data that must be computed or estimated for a new design is known from experience. In consequence many of the steps that require much time in developing a new design can be greatly reduced in length or can be omitted entirely in a revision. Thus the estimation of the weight and the location of the center of gravity of a new design are not only difficult and tedious, but the results are always of doubtful accuracy. In the revision of an existing design, however, only the changes need be considered, and as these normally affect only a small part of the whole, a very accurate weight estimate can usually be made with relatively little labor.

The size of the airplane also has an influence on the design procedure. In a small airplane it is often cheaper to build as the design is being worked out, changing the structure when necessary, than to attempt to work out the design in detail on the drawing board first. For large airplanes, however, or for designs that must be worked out to get the very maximum possible performance, the reverse is the case.

In this chapter, the procedure described will be, in the main, that which is suitable for an entirely new large design, as it is easy to visualize most of the shortcuts that can be taken advantage of for the smaller designs and revisions. In deciding on the advisability of taking any such shortcut, the designer should be satisfied that he has sufficient data and experience to permit him to do so safely, or that there is little chance that he will thereby make any errors, seriously affecting the success of the entire design, which might have been avoided if the longer and more complete procedure had been carried through.

1 : 2. The Specification — The first step in the design of a new airplane is the formulation of a specification or statement of what is desired. For this specification to be satisfactory it should cover at least the following matters:

1. Performance; such as high speed at ground, landing speed, service ceiling, rate of climb at the ground, cruising range, endurance, etc.
2. Disposable or useful load; including number of crew and passengers, weight and, if possible, volume of freight, mail, or express to be carried, weight of fuel and oil, and lists of all equipment to be carried.
3. Required structural strength expressed in terms of required loads and load factors.
4. Any specific requirements regarding details that it is desired to incorporate in the design; such as types of construction to be used, limiting dimensions, desired arrangement of parts of the disposable load, the power plant to be used, etc.
5. Any miscellaneous general requirements regarding stability, maneuverability, cost of construction, etc.

In commercial work the specification given the designer will seldom cover all of these points, whereas the specifications used by the Army and Navy are exceedingly complete and detailed. When the designer is not given a complete specification, it is well for him to elaborate on what is given to him so he will have a definite goal toward which to work.

1 : 3. Singleness of Function — Every airplane is designed for some particular purpose, and the more limited this purpose is made, the greater is the likelihood of its successful attainment. This should be borne in mind by the designer and everything should be subordinated to the attainment of the purpose desired. It is always a great temptation to try to work out a design that will be suitable for two or more different functions, but it is a temptation which should be resisted, as the result is usually an airplane which is not really satisfactory for

any of the purposes for which it was intended. If an all-purpose airplane is desired, it should be called for in the specification, but with the understanding that it will not be as satisfactory for any desired function as an airplane designed solely for that function. After an airplane has been designed and built it will often be found satisfactory for some other use, either as designed or with minor modifications, but only one function should be in mind during the original design.

1 : 4. Selection of Power Plant — The first step after the formulation of the specification, is the selection of the engine to be used. Very often this will be covered in the specification, or will be determined by considerations of price, procurability, etc. If the engine to be used is not definitely indicated by such considerations alone, it will be necessary to make comparative performance computations for airplanes designed around the various power plants between which the choice must be made. The precision required in these comparative estimates will depend upon the narrowness of the margin of merit between the more promising engines for the particular problems.

The final decision will seldom depend solely on the results of the comparative performance estimates. Price, procurability of new engines and parts, reputation for reliability, longevity, and ease of maintenance are all factors that should have weight in the final decision. Special requirements of the design or its proposed use may add to the factors to be considered. For example, air-cooled engines are particularly suitable for use in countries where extreme temperatures are encountered, as Alaska and Arizona.

1 : 5. Preliminary Performance Computations — The engine to be used having been selected, either definitely or tentatively, the next step is to work out a reasonable estimate of the gross weight of the airplane. With the exception of the weight of the structure, and a few minor items, the weights of the various parts of the airplane can be computed with considerable precision from the engine chosen and the data in the specification. The weight of fuel and oil can be computed from the characteristics of the engine used and the specified endurance or cruising range. The weight of the engine is known, and that of the remainder of the power plant can usually be estimated closely by comparison with the power plant weights of other designs using the same engine. The crew, equipment, and pay load of passengers, mail, etc., are of specified weight. The weight of the structure, however, can only be estimated unless the design is practically a revision of an older design. Experience has shown that the structural weight of landplanes varies from about 25 to 35 per cent of the total. In the earlier stages of design a sufficiently accurate estimate of gross weight for use

in deciding on the power plant and in similar preliminary computations can be obtained by multiplying the estimated total weight of all items other than the structure by 1.45.

A reasonable value for the gross weight having been obtained in this manner, a performance estimate should be made according to any of the suitable methods described in works on aerodynamics, reasonable assumptions being made for the propeller efficiency, wing loading, parasite resistance, and any other quantities not yet fully determined.

The accuracy of this performance estimate will depend mainly on the ability of the designer to make the proper assumptions, and therefore on his experience and judgment. The estimate should be sufficiently accurate, however, to settle the question of which of two or more engines under consideration should be used and to give a reasonably correct idea of the possibilities of a design with the useful load and power plant that are planned. It will also give the experienced designer a good idea of the degree of care he must exercise in holding down the weight and parasite resistance if he is to meet or excel the specified performance.

1 : 6. Weight Control — One of the most common reasons for completed airplanes failing to have the expected performance is that their actual weight is in excess of that used in the original performance estimates. This result can be avoided only by the assumption of a reasonable weight in the first place, and by continued study and care to make sure that this estimate shall not be exceeded in construction. For most of the items, the weight is known or can be computed with fair precision. The only important source of error is in the estimation of the weight of the structure, and the problem of weight control is almost entirely one regarding that part of the airplane. In the early stages of design, the designer can do little more than guess the weights of the structural parts, but this should not deter him from making an estimate and holding to it as closely as possible.

Chapter XV describes the recommended method of weight control by "budgeting" and also gives data of considerable value in the making of weight estimates. The point to be emphasized here is that a detailed weight estimate should be made at the beginning of the design process, and that it should be revised and kept up to date by checking it against the actual weights of the various items as the work progresses, every effort being made to prevent an increase in the total weight.

1 : 7. Preliminary Design of Wings — The important characteristics to be determined in the preliminary design of the wings are the number of wings, airfoil section, area, principal dimensions, and type of construction. These things cannot be determined separately, as each

affects the others, and the only way to make a final decision is to make trial performance computations with combinations of the various quantities that the designer believes from his experience will give the desired result.

The number of wings is determined principally by the individual preferences of the designer. The problem is partly aerodynamic and partly structural. The aerodynamic phase of the question will not be gone into here as it is discussed in the text-books on aerodynamics, but the reader is warned that the final choice should depend on the results of a consideration of both phases. The chief structural advantages of the monoplane are that the number of individual parts is normally less than with the biplane, and it is often easier to obtain the desired fields of vision for the crew and passengers. Its disadvantages are the great span which usually involves extra weight and increased hangar space per airplane, and a decrease in the torsional stiffness of the wing structure which may involve danger of wing flutter or lack of lateral control. The advantages and disadvantages of biplanes are the reverse of those of monoplanes. In a few designs, triplanes have been used, particularly for airplanes of large size where it is considered important to reduce the wing span as much as possible, both to obtain a design of minimum structural weight and to reduce the required hangar space.

The choice of airfoils to be used in the trial designs can usually be cut down to two or three by the application of various aerodynamic criteria that will not be described in this volume as they are adequately treated in works on aerodynamics. In deciding upon the dimensions and types of construction to be used in the trial designs, care must be taken to assume data that are suited both to the airfoils and to the function of the airplane. This is a matter in which the designer must rely for guidance mainly on experience. It is one for which no general rules have yet been promulgated.

The total wing area will be determined by the gross weight of the airplane, the airfoil used, and the required landing speed. The weight of the wings, however, and therefore that of the airplane, will depend to some extent on the wing dimensions and the type of construction used, and the designer will have to allow for that fact, though the errors in relative performances computed for the trial designs are not likely to be large on that account if reasonable assumptions are made.

The aspect ratio chosen must result from a compromise between aerodynamic and structural considerations. A large aspect ratio wing has better aerodynamic properties than one with a small aspect ratio. On the other hand, the structural weight of the large aspect ratio wing

will be greater for two reasons. First, the bending moment on the wing structure as a whole is increased, as the load on the wing remains the same but the center of pressure on each side is further from the fuselage. Second, the wing chord being decreased, there is less available depth for the spars, which will usually be heavier as a result.

The span and chord of the individual wings are fairly well determined when the number of wings, aspect ratio, and wing area have been decided on. Minor revisions in these values will usually be necessary during the course of design in order to provide better vision, improve the structural design, etc.

The type of bracing to be used depends mainly on the aspect ratio and the airfoil section. If the aspect ratio is low, the wing may be internally braced. As the aspect ratio increases, a point is reached where the internally braced structure becomes unduly heavy, and a single bay externally braced design becomes more economical. As the aspect ratio increases further, the number of bays must be increased. The aspect ratios forming the limits of economical use of the various numbers of bays have never been determined. Since the exact points of demarcation depend on the design load factor employed, the load per square foot of wing, the airfoil section used, and the type of construction employed, a general solution of the problem is not immediately apparent.

1 : 8. Preliminary Side View — After the designer has decided upon the principal wing dimensions and has worked out a preliminary detailed weight estimate, he is ready to begin work on the detailed design. The first drawing to be made is a side elevation. The function of the fuselage being to carry the engine, crew, equipment, etc., the more important and bulkier of these items are first located in their desired relative positions, and the fuselage structure and outline added. The order in which the contents of the fuselage should be located depends primarily upon the function for which the airplane is designed. In a commercial airplane the location of the passengers, mail, or express would form the starting point, around which the pilot's cockpit, engine, etc., would be grouped. In a military airplane the pilot, guns and bombs would be located first, and the other items later. The preliminary design of the fuselage should be made with care to put in a good place every item that is heavy or difficult to move without impairing the suitability of the airplane for its function. Other items may be located with less care, but all of the contents of the fuselage should be located on the drawing. At this stage of the design process it is desirable to begin the construction of a "mock-up," such as is described in Article 1 : 14.

After the contents have been located, and the forward portion of the fuselage outlined, the empennage and landing gear can be designed roughly, and these units and the remainder of the fuselage drawn in. The location and size of the empennage is decided upon by considerations of stability and maneuverability that are discussed in works on aerodynamics.[1]

At this stage the important item of landing gear design is the location of the axle in side view. The size of wheels is determined by the gross weight of the airplane. They are located so that three important conditions are fulfilled. When the airplane is taxiing along the ground with the thrust line horizontal, the propeller must clear the ground by at least 9 inches. In the case of seaplanes, the propeller clearance above the water should be at least 18 inches. At the same time the angle between the ground and a line tangent to the wheels and the tail skid, both in their fully deflected positions, should be about 13 degrees. This is to allow the wings to have the angle of attack of maximum lift coefficient, or nearly that, when taxiing with the tail down. The third condition is that the axle should be far enough forward to minimize the danger of nosing over in landing. This is generally attained by making the angle in side view between a perpendicular to the thrust line and a line joining the axle with the center of gravity of the airplane not less than 12 degrees. This angle should not be made too large, however, as an airplane with the center of gravity too far behind the axle has a tendency to "ground loop" when brakes are not used.

After the empennage and landing gear, the cross-section of the wings is added to the drawing. Their airfoil section and chord are assumed to have been determined previously in a preliminary performance estimate; their location can be settled only after making a balance computation.

1 : 9. Preliminary Balance Computation — After the preliminary side view has been completed, except for locating the wings, a preliminary balance schedule should be compiled. This schedule should list each item of the airplane's weight, except those of the wings and their contents, with the weight and location of the center of gravity of each. The centers of gravity are usually located by their distances from two mutually perpendicular lines. Any two such lines could be used for these "axes of reference" but it is customary and convenient to take the propeller axis or thrust line as one of these lines. The location of the "origin" or intersection of the axes of reference is not so well

[1] L. Bairstow, "Applied Aerodynamics." W. S. Diehl, "Engineering Aerodynamics." E. P. Warner, "Airplane Design." E. B. Wilson, "Aeronautics."

standardized. The intersection of the thrust line with the front face of the rear propeller flange is often used but when one fuselage is to be used with several engine mounts and engine installations it is better to take a definite point in the fuselage structure such as would be obtained if the vertical axis passed through the first panel point behind the engine mount attachment points. With the weights and their locations listed, it is a simple matter to compute the moment of each item about the selected origin and determine the location of the center of gravity. The balance schedule made out as outlined above is for the fuselage and contents only. For a complete balance schedule and the determination of the center of gravity of the whole airplane, it is necessary to extend this schedule by adding items representing the wings and any other parts not already taken care of. This cannot be done at first as the location of the wings depends, as is shown in the next article, on the location of the center of gravity of the remainder of the airplane, but should be done as soon as the wings have been tentatively located.

In determining the center of gravity locations for the final design and balance computations, the distances of the centroids of the constituent parts of the airplane above or below the origin should be determined so that the vertical location of the center of gravity may be obtained, but in the early stages of design the horizontal location or more correctly the location along the thrust line is sufficient.

1 : 10. Location of Wings — The two most important factors influencing the location of the wings are the effect of the wing location upon the flying characteristics of the airplane, and the problem of vision. The flying characteristics depend on the relative location of the center of gravity of the airplane and the mean aerodynamic chord of the wings (usually abbreviated M.A.C.). The effect of the movement of the center of gravity with respect to the M.A.C. is discussed in books on aerodynamics in their treatment of stability and will not be gone into here. A brief consideration of the problem of vision is given in Art. 1 : 11.

The cross-section of the wings is first sketched in on the side elevation so that the center of gravity of the airplane will be in the desired location with respect to the M.A.C. For a first trial it is close enough to place the wings so that the center of gravity of the fuselage and contents as obtained from the preliminary balance schedule is at 30 per cent of the M.A.C. The balance schedule can then be extended to include the wings and give the center of gravity location for the complete airplane. As further changes are made in the estimated weights or their locations, their effect on the location of the center of gravity of the

whole airplane would be studied by making proper corrections to the complete balance schedule.

Often it will be found that the first position chosen will be poor from the standpoint of vision. Three methods are available to the designer to correct this defect, and he will usually make use of a combination of all three. The arrangement of the wings can be changed by the use of stagger or sweepback, changing the relative chords and areas, etc.; the trailing edge portion of the wings may be cut away; or the position of the contents of the fuselage may be rearranged to shift the relative location of the center of gravity with respect to the M.A.C. As the engine is one of the heaviest of the fuselage contents, it is desirable, when drawing the preliminary side view, to place it so it can be moved backward or forward a few inches to help obtain the desired balance.

Since the aerodynamic properties of the wings finally selected may differ from those of the wings used in the preliminary performance estimate, and since the design is now taking definite shape, the performance, weight and balance computations should be revised at this stage. The revised figures are likely to show that additional changes are needed, but, by a process of trial and error, a satisfactory arrangement will finally be obtained.

1 : 11. Vision — The vision requirements of all types of airplanes are exceedingly important. They vary greatly with the function of the airplane but in any case they must be given careful consideration by the designer. They furnish one of the main reasons why it is so difficult to design an airplane that will be satisfactory for more than one purpose. The minimum vision requirements for the various military types are outlined in the various Army type specifications, but there are as yet no standardized requirements on commercial designs. Certain general principles and methods of obtaining better vision, however, apply to all types, and they will be discussed briefly.

The pilot must always have a field of view which will permit him to fly his airplane and land it with ease and safety. In military designs, when the crew consists of three or more persons, a judicious arrangement of the personnel will relieve the pilot of all duties except that of flying the airplane, and a careful arrangement of the remaining members of the crew should leave no blind spots which would permit an enemy aircraft to approach unseen. As the number of the crew is reduced, the importance of a wide field of vision for each individual member is increased, and in a single-seater pursuit, where the pilot must also assume the duties of observer, visibility attains its maximum importance and can be secured only by giving the greatest care and attention to

the design of the structure as a whole. In commercial planes there is no essential need of visibility for anyone except the pilot, although it is desirable to give the passengers on transports as much of a view as possible of the country being traversed.

If the pilot has considerable freedom of movement, he may, by changing his position, look around almost any obstruction and reduce the blind spots to a minimum. Ordinarily, however, he is strapped in his seat with a belt so that his movements are restricted to bending at the waist and neck, and turning the head. The cockpit should be designed to enable the pilot to take the maximum advantage of such movements as are possible to improve his field of view. From a properly designed open cockpit, the pilot should be able to look over the side and see directly beneath the fuselage. With monocoque construction it may prove impracticable to cut out a large enough opening without seriously weakening the structure, but with a conventional steel tube truss fuselage, the cowling can be cut away as far as the top longeron without any difficulty.

Many of the new designs, particularly the larger ones, have the pilot enclosed in a cabin. This usually prevents him from looking down over the side of the fuselage but greatly reduces the physical strain of flying, at least in daylight. In cases where the pilot's cabin can be placed in front of the wings, as in pushers and many multi-motored designs, the decreased field of vision over the sides of the fuselage can be more than compensated for by the absence of wings obstructing his vision in landing. When such cabins are used, care should be taken that there are sufficient windows and that none of the structural parts of the fuselage are in locations that seriously affect the pilot's vision. Numerous windows, however, may involve cross reflections in night flying and increase the strain on the pilot.

A narrow fuselage, especially at the engine section of a tractor airplane, is favorable to a good field of view. It is difficult to secure good vision about the wide blunt nose of a radial motor housing, while the narrow section possible where the cylinders of the engine are vertical and in line creates a very good field of view forward. A large nose radiator increases the blind area in front, the most favorable condition being reached when the engine cowling can be run down to an approximately conical shape with the point at the hub of the propeller. While it is desirable to have an engine housing which will protect the engine from the weather as much as possible, the visibility can be greatly improved by reducing to a minimum the volume enclosed by the cowling. The cylinder heads may be left exposed, and along the sides of a vertical motor the cowling may be "dished in" to improve the field of view.

To secure good visibility about a wing, it is desirable that the pilot be so placed that his eyes are nearly on the line of the chord. Perfect visibility can be secured with a parasol monoplane, or about the upper and middle wings of a biplane and triplane respectively. In a biplane if the trailing edge of the center section is cut away to enable the pilot to be properly placed, he can uncover all the blind area obstructed by the wing unless it is of extremely thick section. The narrow chord of the wings of a triplane tends to reduce to a minimum the blind area obstructed by them. Good visibility past the lower wing of a biplane is more difficult to secure. It is impossible to eliminate this blind spot entirely, but it may be cut to a minimum by careful design. The use of a narrower chord on the lower wing than on the upper, a wide gap which will place the pilot high above the lower wing, a smaller span to the lower than to the upper wing, and the cutting away of the trailing edge near the fuselage are devices at the disposal of the designer which will cut the blind area to a minimum. The use of a tapering wing with wide chord at the fuselage and the use of large dihedral in the lower wing are both factors which tend to increase the blind area under the wing.

Stagger is an arrangement which greatly assists the designer in securing visibility by enabling him to locate the wings in the most advantageous position with relation to the pilot. A slight stagger is often necessary if the pilot of a single-seater biplane is to be placed with his eye in line with the chord of the upper wing. By use of stagger the lower wing may be placed so that the area obstructed by it is in the least important part of the field of view. A maximum angle of vision over the leading edge of the lower wing is, in general, more important than a large angle over the trailing edge, and an adjustment of the stagger may be used to place the lower wing in the position which gives the best results.

External obstructions such as struts, exhaust manifolds, radiators, etc., interfere with the pilot's view to a greater or less extent. Where the obstructing surfaces are small and detached from larger obstructing surfaces, as is the case with narrow struts, a very slight movement of the pilot's head will enable him to see all the area behind them so that their effect on the field of view is unimportant. But in the case of radiators, and exhaust manifolds which are directly attached to a large obstructing surface, the field of view may be materially affected. The use of short exhaust stacks on V-type engines and the housing of the guns of military airplanes inside the fuselage are desirable, while the building of the radiator into the wing so that it conforms to the wing curve undoubtedly gives the best installation from the visibility standpoint.

1 : 12. The Three-view Drawing — After a satisfactory side view has been completed the designer should draw in the front elevation and the plan view to complete the three-view drawing. In order to do this he must decide upon the size and shape of the control surfaces, and the general design of the landing gear and interplane bracing.

The three-view drawing is the basis for the detailed design of the airplane and all of the other drawings may be considered as close-ups of parts of it. The drawing, however, should show as much detail as is consistent with clearness. A drawing which shows only the bare outlines of the external shape of the airplane may be satisfactory for the construction of a wind tunnel model, but is not very satisfactory for many of the uses for which a three-view drawing is desired. While the three-view drawing is spoken of as a single drawing, it is not at all essential that all three views be put on one piece of paper, and it is often more convenient to have each view on a separate sheet.

As the process of design is carried on, many modifications will be made in the three-view drawing. It should therefore be made on a good grade of heavy paper which will permit numerous erasures and changes.

1 : 13. The Wind Tunnel Model — As soon as the three-view drawing has been completed sufficiently to determine the external shape of the airplane, a wind tunnel model may be constructed. The requirements for this model are discussed in works on aerodynamics. The wind tunnel model tests will usually show that certain modifications in the original design are desirable if not necessary. When possible, the model should be reworked to agree with the changes in the design and tested again to make sure that the changes will have the desired effect. This is not very often done on account of the cost, but is a very desirable procedure. Many designers have a tendency to delay the construction and test of the wind tunnel model until the detailed design has been completed in order to have the model as much like the finished airplane as possible. This is not good practice, however, as it is then much harder, if not impossible, to make the changes that the test shows to be desirable, whereas if the model is tested during the early stages of the design, such changes can be made without great difficulty. Sometimes the wind tunnel model will show that the design has no chance of meeting the required performance and that a thorough overhauling and revision of the original design is necessary. Since that is the case it is evidently best to obtain the tests as soon as possible.

Often no wind tunnel model is made. When the new design is a revision of an existing design, it may be unnecessary as the flight test data from the earlier airplane may be corrected, to allow for the changes

being made, with less error than is involved in the use of a model. Furthermore, the model, if it is to be of any use, must be very carefully made and will be so expensive to construct that for a small inexpensive type of design it may actually be cheaper to build the airplane and try it out with different sets of wings, or with the wings in different locations. This should not be attempted, however, unless the designer has sufficient experience to be sure that his first attempt will have reasonably good flying characteristics. For large expensive types, a model should always be made and tested, as the cut and try process is not likely to be economical.

A wind tunnel model should always be tested whenever the two wings are of different airfoils, have decalage, or excessive stagger, or other large deviations from conventional design, since in such cases the test data are needed to determine the relative distribution of load on the wings and the balance characteristics of the design.

1 : 14. The Mock-up — In conventional designs, especially those with a single engine, nearly all of the furnishings, equipment, armament, etc., are located with the power plant, crew, and passengers or cargo in the forward part of the fuselage. Owing to the number of items that must be placed in this part of the structure, and the necessity of economizing space, it is practically impossible to locate all the various items by drawings alone. The device usually employed in new designs to facilitate the proper location of the fuselage contents is the construction of a mock-up or full scale model of that part of the airplane. Usually the mock-up represents the fuselage from the rear of the cabin or rear cockpit forward, and includes the center section of the wings. In special cases it may represent more or less of the structure. In the mock-up, the structural members are often represented by members of rough wood, and the items of power plant, equipment, etc., either by the actual articles to be used, or by dummies correct to scale. Experience has proved that the use of dummies is likely to be very unsatisfactory owing to the difficulty of having them correct.

The mock-up not only aids in the location without mutual interference, of the things that must go into the fuselage, but also is a great help in determining the qualities of the design with respect to vision, comfort of the pilot, accessibility of the power plant, equipment, armament, etc.

The actual structure of the fuselage and center section may be used in place of a mock-up, particularly when revising an existing design, the final location of the various contents of the fuselage being determined after the main structure has been built. This practice, however, may lead to difficulties, as the mock-up may show that it is desirable

to change the location of some of the structural members. The possibility of such changes, however, depends largely upon the skill and experience of the designer, and some very successful airplanes have been produced in this manner.

1 : 15. Detailed Design — After the completion of the wind tunnel tests and the mock-up, the final detailed design can proceed without delay. As the detailed design proceeds, it will often be found necessary or desirable to change the design from that implied in the preliminary work. Whenever this is done, the effect of such changes on all of the other factors must be considered.

Some designers prefer to make complete drawings of every part before it is constructed and assembled in the airplane. Others make preliminary drawings for only the main parts of the structure, working out the details as they are put together in assembly. When the former method is used care must be exercised to see that the drawings are corrected for any changes that are found necessary in putting the airplane together. With the latter method, it is necessary to make drawings of the various parts after they have been constructed if any more airplanes of the same design are to be built, and in any case, enough to permit the Department of Commerce to check the strength of the design. In this method, the first airplane built has the characteristics of a very complete mock-up, except that it is entirely made of actual parts, and can be flown after it has been completed.

In the preceding articles, the process of design has been described as one with a rather definite sequence of operations. While this is true as to the more general lines of the design process, it is not true in detail. Much of the work which is described as being carried out in series may be carried out in parallel, and the various steps may overlap or be carried out in a different order from that described. The impression may also have been given that the designer will be thinking of but one phase of his design at a time. This is far from being the case. The truth is that he must be considering, to some extent, all of the various phases simultaneously, nearly all of the time. He must also be prepared to make constant revisions of the work that has once been accomplished. If this is not done, the final design is almost certain to vary greatly from the estimated weight, and in all probability the performance will be below that specified.

CHAPTER II

THE CRITICAL LOADING CONDITIONS

Every airplane must be provided with a structure designed to withstand the most serious of the infinite number of possible combinations of external forces that may act on it in flight and landing. No single combination will be the most serious for all parts of the structure, so it is necessary to design for a system of loading conditions which will include the critical loadings for all of them. We do not yet know enough about aerodynamics to predict with certainty the worst loads that will be imposed on any airplane, but we can do it within reasonably close limits, and have found that airplane structures designed for a few standard loading conditions prove satisfactory in service. In this chapter the conditions of flight and landing represented by these standard loading conditions are described and reasons are given why they are considered critical. No attempt is made to go into the details of the various conditions, but enough is said to indicate why they are critical and why the rules for computing the external loads on the airplane promulgated by the Army, Navy, and Department of Commerce are reasonable in their general outlines. It would be interesting to go into the details of those rules and consider the justification for the various arbitrary assumptions and empirical formulas that are used where sufficient theoretical data are lacking,[1] but limitations of space do not permit.

2 : 1. Character of Forces Acting on an Airplane — The forces that must be considered in the design of an airplane structure are the weight and inertia forces of the airplane and its contents, the air pressures, the propeller thrust, the engine torque forces, and the ground reactions.

The weight is that constant force, proportional to the mass of the body, which tends to draw every physical body towards the center of the earth. The air pressures are the forces imposed by the resistance of the air to the passage of the airplane through it. The designer should be familiar with these forces from his study of aerodynamics. The propeller thrust, generated by the power plant, is the force used to pull the airplane through the air, while the engine torque is an accompanying force of no direct utility which cannot be avoided with the

[1] Such a detailed discussion of the rules is given in Chapter VI of "Airplane Design," Vol. I, by A. S. Niles.

present types of power plant. The ground reactions are the forces imposed on the airplane by the ground or water when landing, taxiing, or at rest. The inertia forces are the reactions to the forces, if any, that produce acceleration of the airplane or its elements.

In order to simplify the discussion, the air pressures are divided into three types: lift, drag, and transverse forces. The lift forces are the components of air pressure parallel to the plane of symmetry and perpendicular to the line of flight. The drag forces are the components parallel to the line of flight, and the transverse forces the components perpendicular to the plane of symmetry of the airplane. As an airplane is approximately symmetrical, the vector sum of the transverse forces is negligible, except in a few of the less important flying conditions. In the others these forces are neglected. The terms "upward" and "downward" are used to describe the directions of the lift and parallel forces with respect to the airplane in the manner that is strictly correct only when the airplane is in normal horizontal flight.

2 : 2. Inertia Forces — When a body is in a condition of equilibrium, i.e., at rest or in unaccelerated motion, the vector or geometrical sum of all the forces acting on it, or any part of it, must be equal to zero. This fact is the basis of the methods used for computing the unknown forces acting on the body or its parts with the aid of the equations of equilibrium. If a body is in accelerated motion, the vector sum of the forces acting on it is not zero, but is a force acting in the direction of the acceleration and equal to the product of the mass of the body and its acceleration. Therefore, if we represent the presence of the acceleration by a hypothetical external force equal to the product of the mass of the body and its acceleration, but acting in a direction opposite to the latter, the vector sum of all the forces acting on the accelerating body will be zero, and the equations of equilibrium may be validly applied to it. The force used to represent the presence of the acceleration in this operation is called the inertia force of the body. This is a statement, in somewhat different terms, of d'Alembert's Principle.

When all parts of a body have the same acceleration — a body accelerating in translation — the inertia forces are distributed in proportion to the masses of the various parts and their resultant will pass through the center of gravity of the body. If the body is rotating with uniform angular velocity, each element will have an acceleration towards the center of rotation which may be represented by an inertia force in the opposite direction. The resultant of the inertia forces due to rotation about an axis passing through the center of gravity of the body will be zero.

The nature of inertia forces and the validity of their use in the equa-

tions of equilibrium to analyze bodies in accelerated motion should be thoroughly understood as they are of utmost importance in airplane design.

2 : 3. The Standard Loading Conditions — The standard loading conditions used in the United States may be divided into three groups: the main flying conditions, the main landing conditions, and the minor loading conditions. The main flying conditions represent possible situations connected with the entry into, maintenance of, and recovery from a dive. Although there are other maneuvers which involve heavy loads on the structure, in nearly all cases, either those loads are less severe than the ones connected with the various phases of the dive or they are imposed when the conditions of flight are practically identical with some phase of diving. The main landing conditions represent the extremes of the range of flight attitudes at which satisfactory landings are made and thus cover the intermediate conditions. The minor loading conditions are devised to ensure adequate strength in certain parts of the structure not critically loaded in any of the main conditions or to take care of unusual situations or design features.

The loading conditions are discussed here in the order in which they are listed above. In the first group the first to be considered is the Nose Dive condition representing a dive at high speed along a straight flight path. This is followed by the High and Low Angle of Attack conditions representing stages of recovery from a dive and any other maneuvers involving a rapid increase in the angle of attack of the wings. Then the Inverted Flight condition representing the entry into a dive and certain other possibilities of flight is considered. The main landing conditions are taken up next, and finally the minor conditions are discussed.

2 : 4. The Nose Dive Condition — The forces acting on an airplane in a steep dive are shown in Fig. 2 : 1. Instead of showing the individual weights of the parts of the airplane, they are represented by their resultant passing through the center of gravity, the vertical force, W. This is resolved into two components, $W \sin \theta$ parallel to the line of flight, and $W \cos \theta$ perpendicular to it. The air pressures are represented by four resultant forces; the resultant drag, D; the "downward" lift force, L_f, on the forward portion of the wing; the "upward" lift force, L_r, on the rear portion of the wing; and the resultant lift on the tail, E. The other air pressures are small enough to be neglected. Propeller thrust is neglected as the speed of airplanes in dives of the type being considered is so great that the propeller would no longer be producing thrust, but might even be acting as a brake and producing

drag. Torque is also omitted because it is of negligible importance in determining which of the flying conditions are critical.

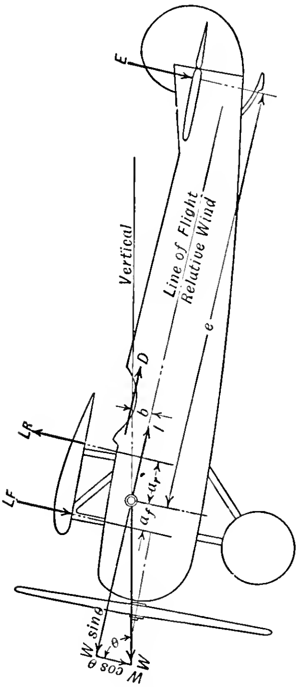


Fig. 2 : 1

Two forces are used to represent the lift on the wings because the more common airfoils, when in a dive at high speed, are subjected to "downward" air pressure on their forward and "upward" pressure on their rear portions. These two forces could be replaced by their resultant, but that would be a relatively small force acting at quite a distance from the actual wing, and this would complicate rather than clarify the discussion.

It can be seen from the figure that the forward motion of the airplane is aided by the weight component $W \sin \theta$ and resisted by the drag, D. If the former is the larger, part of it will be unbalanced and will produce the acceleration and the accompanying inertia force, I, at the center of gravity. No other inertia forces need be considered as the airplane is assumed to be diving along a straight path.

If we apply the conditions of equilibrium we obtain the following equations, the X and Y axes being assumed parallel and perpendicular respectively to the line of flight, and the center of moments at the center of gravity.

$$\Sigma X = 0; \quad W \sin \theta = D + I \qquad 2:1$$

$$\Sigma Y = 0; \quad W \cos \theta = L_r - L_f - E \qquad 2:2$$

$$\Sigma M = 0; \quad L_f a_f + L_r a_r = D \times b + E \times e \qquad 2:3$$

Since the air pressures, L_f, L_r, and D at any given angle of attack of the wings are proportional to the square of the velocity, the maximum forces will be present when the speed is a maximum. Furthermore we can substitute $\rho C_{DA} A V^2$ for D in equation 2 : 1 and write it as follows:

$$W \sin \theta = \rho C_{DA} A V^2 + I \qquad 2:4$$

The quantity C_{DA} is one which may be called the drag coefficient of the airplane as a whole, its magnitude being whatever is needed to

make the expression $\rho C_{DA} A V^2$ equal to the total drag when ρ is the mass density of the air, A the area of the wings, and V the velocity.

From equation 2 : 4 it appears that for a given angle of flight path, θ, the maximum velocity will be obtained when the angle of attack is that of minimum C_{DA} and the airplane has speeded up until I has become zero. From this equation alone it would seem that the maximum possible speed or "terminal velocity" would be obtained when the flight path is vertical as then sin θ would be a maximum. This is not the case, however, as the other two equations of equilibrium must also be satisfied. At the angle of attack of minimum C_{DA} the moment of L_f and L_r about the center of gravity is so great that the tail load E will be larger than their resultant. Hence in order to satisfy equation 2 : 2, W cos θ must be greater than zero and θ cannot be quite equal to 90 degrees though it may approach that value quite closely.

Wind tunnel tests show that, with the airfoils in common use, the individual values of L_f and L_r will be quite large when the airplane is diving at high speed and the angle of attack of the wings is that of minimum C_{DA}. The result is a strong tendency to twist the wings which often makes the Nose Dive condition the critical one for the rear lower spar, incidence bracing, some members of the lower drag truss, and some members of the fuselage near its connections to the wings. The stresses developed in other parts of the structure are not likely to be critical, so the stress analysis for this condition is usually limited to the wings and adjacent portions of the fuselage.

Since pilots seldom if ever permit their airplanes to speed up until the terminal velocity is reached, particularly with the larger and less maneuverable airplanes, and since the unknown contribution of the propeller to the drag makes it impossible to determine the true terminal velocity of a given design, it has been found impracticable, as yet, to formulate satisfactory rules based on theory for the Nose Dive condition. As a result, the specifications in use provide for a system of loading similar to that shown in Fig. 2 : 1 but with numerical values obtained from rather arbitrary empirical rules. It is only reasonable to expect, however, that as research continues the loading rules for the Nose Dive condition will eventually be put on at least as sound a theoretical basis as those for the High Angle of Attack condition.

2 : 5. The Recovery from a Dive — If the angle of attack of an airplane flying at constant speed along a straight path is increased, from any cause, the character of the flight path and the loads acting on the airplane will be greatly changed. This change in angle of attack may be caused by manipulation of the controls, or by flying through "bumpy" air. Before the change in angle of attack there are no inertia forces

acting and the vector sum of the weight, air pressures, and propeller thrust will be zero. The increase in the angle of attack, no matter how caused, will involve an increase in the lift and drag coefficients and usually a forward movement of the center of pressure. The increase in the drag coefficient will cause an increase in the total drag and produce a reduction in velocity that will be accompanied by a corresponding inertia force with the more common airfoils, the increase in the lift coefficient will be faster than the decrease in V^2 due to the increase in drag, so the resultant lift on the wings will be increased. The added lift load will give the airplane an "upward" acceleration, that will be accompanied by a "downward" inertia force. As this acceleration is at right angles to the line of flight it will cause the direction of the velocity of the airplane as a whole to be constantly changing, that is, the line of flight will be a curve. Thus the airplane will "zoom" if originally flying along a horizontal path or will come out of a dive or glide if originally flying along an inclined path. The changes in the magnitudes, and usually of the points of application, of the air loads on the wings will change the lift load on the tail required to satisfy the criterion of $\Sigma M = 0$. They will also cause the airplane to rotate about its center of gravity producing acceleration accompanied by inertia forces. In practice this effect is neglected, as what data are available indicate that these inertia forces are small in comparison with the other inertia forces, and that their neglect is normally on the safe side.

At angles of attack approaching that of maximum lift coefficient, the forces acting will be about as shown in Fig. 2 : 2, in which it is assumed that the airplane has been pulled out of a dive or horizontal flight and is now climbing. If the reverse assumption had been made, the figure and the resulting equations of equilibrium would have to be modified, but the changes should not be difficult to make, and in any event would not affect the major results arrived at in the following discussion.

The force system of Fig. 2 : 2 shows several important changes from that of Fig. 2 : 1. The weight is still shown as a vertical force acting at the center of gravity and divided into two components. In this figure, however, as θ is a relatively small angle, $W \cos \theta$, the component opposed to the lift, is nearly equal to W, while $W \sin \theta$, the component parallel to the drag, is rather small and assists the drag instead of opposing it. Instead of showing the lift on the wings as two forces, L_f and L_r, it is represented by a single force, L, acting at the center of pressure. This is desirable as the lift components along the entire chord act in the same direction at the larger angles of attack, and the center of pressure is on the wing, so that the special reasons for using

two forces in Fig. 2 : 1 do not exist. The drag, D, is shown very much as in Fig. 2 : 1, but its magnitude must be considered to have changed, and as the change is localized on the wings, its point of application must be considered to have moved. The tail load, E, is shown as an "up" load as we are considering a condition in which the center of pressure is near its most forward position, and an "up" load on the tail will probably be needed for equilibrium. The existence of the acceleration is represented by two inertia forces, I_L and I_D, acting at the center of gravity and parallel to the lift and drag respectively. The accelerations due to rotation are neglected for the reasons stated above.

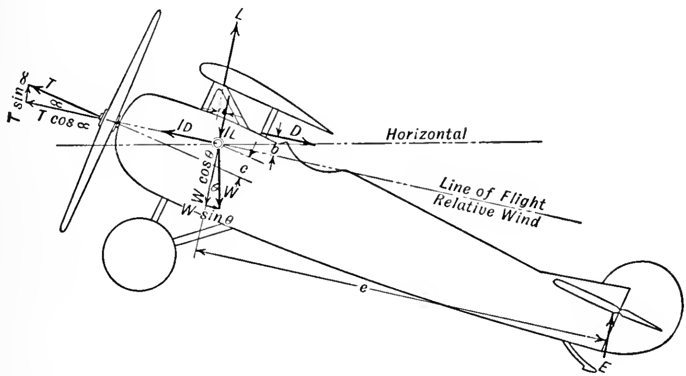


FIG. 2 : 2

The propeller thrust, T, is shown, though the speed may be so great that it may be negligible or acting in the opposite direction. It is divided into two components, as shown, since it will usually act at a small angle to the line of flight.

Applying the conditions of equilibrium to the forces of Fig. 2 : 2 as they were applied to Fig. 2 : 1, we obtain the following equations:

$$\Sigma X = 0;\ D + W \sin\theta = T\cos\alpha + I_D \qquad 2:5$$
$$\Sigma Y = 0;\ L + T\sin\alpha + E = W\cos\theta + I_L \qquad 2:6$$
$$\Sigma M = 0;\ L \times a + D \times b + T \times c = E \times e \qquad 2:7$$

For any given velocity, V, the lift, L, will be equal to $\rho C_L A V^2$ and its maximum possible value for that velocity will be $\rho C_{L\max} A V^2$. But, if V_s is the stalling speed, or minimum velocity at which horizontal flight can be maintained, W equals $\rho C_{L\max} A V_s^2$. Combining these relations we obtain the very important formula

$$\frac{L_{\max}}{W} = \frac{V^2}{V_s^2} \qquad 2:8$$

TABLE 2 : 1

Load Factors Recorded in Flight

Maneuver	Approx. M.P.H.	Training					Observation				Pursuit				Bomber			Cargo	
		JN-6H	PT-1 180 H.P.	PT-1 150 H.P.	TW-5	VE-9	CO-4	XCO-6	XO-2	XO-1	MB-3A	PW-7	PW-8	PW-9	NBS-1	NBS-4	DT	XA-1	T-3
Sharp pull-out	46–65	2.3	2.1	2.1	1.5	1.9	1.7	...	2.7	...	...	...	...	...	1.0	1.5	1.6	1.4	1.6
	66–85	3.3	3.0	2.7	2.2	2.6	2.6	2.0	3.2	...	2.5	1.6	2.0	2.2	2.0	2.0	2.5	2.1	2.6
	86–105	...	4.2	3.9	3.2	4.0	3.3	3.4	3.3	...	3.5	2.7	3.7	3.2	3.1	3.0	2.9	2.9	2.8
	106–125	...	5.2	4.3	4.7	5.0	4.2	4.2	4.2	...	5.0	3.9	5.0	3.9	...	...	...	...	3.3
	126–145	...	...	...	...	...	...	4.4	...	3.8	...	5.3	5.8	5.0	...	...	...	...	...
	146–165	...	...	...	...	...	...	4.5	...	4.1	...	6.4	7.0	6.4	...	...	...	...	...
	166–185	...	...	...	...	...	...	...	...	5.4	...	7.8	...	...	...	...	...	...	...
Gradual pull-out	66–85	...	...	...	...	...	...	1.8	...	...	...	...	...	...	...	...	...	...	...
	86–105	2.1	...	1.9	...	...	...	2.0	...	...	...	...	...	...	1.5	2.3	...	...	...
	106–125	1.8	3.9	1.7	4.1	...	4.1	2.1	2.8	...	...	...	...	...	2.0	2.8	1.9	2.7–2.9	1.6
	126–145	...	4.9	1.9	5.0	2.25–3.4	4.6	2.3	3.3	...	...	...	4.4	4.6	1.5	3.0	2.6	3.6	2.4
	146–165	...	5.5	...	5.7	...	5.4	3.0	3.4	...	...	...	4.8	5.5	...	3.0	...	...	2.9
	166–185	...	...	...	...	...	...	...	3.7	...	...	...	5.6	5.8	...	...	...	...	...
	186–205	...	...	...	...	...	...	...	...	...	...	...	7.0	7.3	...	...	...	...	...
Wide loop		2.3	2.7	1.8	2.3	3.8	3.3	2.2	3.2	3.25	1.6	2.7	3.2	3.0	...	...	...	...	...
Tight loop		3.8	3.6	2.2	3.7	4.4	3.3	3.1	3.8	...	2.7	6.1	4.25	4.9	...	...	...	...	...
Tail spin, power off		3.1	2.0	2.0	2.9	2.5	2.4*	2.8	3.0	...	2.0	2.6	3.6	5.8	...	...	...	2.4*	...
Sharp 180 degree turn (Vertical Bank)		1.85	2.8	2.1	3.8	3.2	2.9	2.2	3.6	3.5	3.7	5.7	4.0	4.25	2.9	1.6	2.0	2.5	2.9
Tail spin, power on		3.3	2.0	1.7	3.3	2.6	3.4*	2.7	2.6	3.3	2.6	2.3	4.0	5.1	...	...	...	2.0	...
Barrel roll		3.5	4.2	3.2	4.2	3.0**	4.1*	3.9	3.5	4.5	4.7	7.2	6.0	5.2	...	...	...	...	...
Immelman turn		...	...	...	...	...	...	...	...	...	...	4.4	...	...	...	...	...	...	...
Spiral		...	...	...	...	1.4	...	...	...	...	...	5.5	...	...	...	...	...	...	...
Flying on back		...	...	...	...	...	...	...	...	...	...	1.3	...	...	...	...	...	...	...
Mock fight		...	...	...	...	...	...	...	...	...	...	...	...	5.0	...	...	...	...	...
Landing		2.9	2.1	3.0	4.4	2.3	2.0	2.7	1.6	1.6	2.8	...	4.0	2.3	2.6	1.9	2.6	2.0	2.0
Take off		3.0	1.9	2.0	2.6	1.8	2.0	2.5	1.7	1.9	2.0	...	2.9	2.9	2.0	2.0	2.1	1.5	2.0

* Maneuver attempted, but could not be accomplished.
** Half Roll.

Thus the maximum lift load that can be imposed on the wings at any given velocity of flight is equal to the weight of the airplane multiplied by the square of the ratio of that velocity to V_s. The maximum conceivable lift load that could be imposed on the wings would be that produced if the angle of attack were increased to that of $C_{L\max}$ while the speed of the airplane remained equal to the terminal velocity in a dive. Although it is not possible to compute V_t with precision, it is known that this load would be about 15 W to 20 W for most designs, and if the drag produced by the propeller is small may be even larger. Fortunately the maneuver which would produce this load is not one that must be designed for as it is avoided by pilots, both on account of the practical certainty of breaking the structure, and the unpleasant physical effects on the pilot of maneuvers involving heavy lift loads on the wings. What must be designed for are the maneuvers likely to be carried out by the pilot and the sudden changes in angle of attack caused by flying through bumpy air. The severity of the loads produced by the intended maneuvers varies greatly with the size and maneuverability of the airplane, and the uses to which it is put, while the loads caused by flying in bumpy air depend mainly on the ratio of high speed at the ground to stalling speed. Both of these factors are considered in the rules for determining the loads for which a given airplane structure must be designed.

The best data that are available for determining the loads to which the wings of an airplane may be subjected in practice are the results of flight tests in which the total loads are actually measured by an accelerometer in terms of the weight of the airplane. The results of a number of such tests made at McCook Field are listed in Table 2 : 1.

2 : 6. High and Low Angles of Attack — In the Nose Dive condition there is a "down" load on the forward portion of the wing and an "up" load on the rear portion. As the angle of attack increases, if the speed remains constant, the "up" load on the rear portion slowly increases. At the same time, however, the "down" load on the forward portion decreases to zero and is replaced by a rapidly increasing "up" load. As a result, the center of pressure moves forward on the conventional type of airfoil until the lift coefficient is about half its maximum value, after which it suffers little change in position until the maximum value is reached. As the largest proportion of the total load on the wing carried by the front spar is imposed on it when the center of pressure is at its most forward position, and the maximum wing load is produced when flying at the angle of $C_{L\max}$, a loading condition in which these two things happened simultaneously would be critical for the front spar. These two phenomena do not necessarily

occur at the same time, but the error involved in assuming that they do is small, and the resulting loading condition is that called High Angle of Attack. In this condition, both the lift loads on the wings and the inertia forces are assumed to be greater than in any other flying condition. It is usually the critical condition for the front lift truss, some members of both drag trusses, and the forward portion of the fuselage.

If we could be certain that the speed of the airplane would remain approximately constant in the various maneuvers it would not be necessary to design the wings for any of the flight attitudes between those represented by the Nose Dive and High Angle of Attack conditions. Though the rear spar would take a smaller proportion of the total wing load as the center of pressure moved forward, it would continue to take a larger absolute load until $C_{L\max}$ was reached and the High Angle condition would represent its critical loading also, except for the negligible error caused by the slight rearward movement of the center of pressure from its most forward position to that at $C_{L\max}$. This assumption of constant speed, however, is not reasonable, as it is probable, particularly in maneuvers started at very high speeds, that the velocity decreases as the angle of attack increases. This decrease in velocity may well involve a decrease in the load on the rear portion of the wing greater than the increase due to the simultaneous change in C_L, so that the maximum load on the rear spar may be imposed when the center of pressure is still considerably to the rear of its most forward position. This possibility is taken care of by the Low Angle of Attack loading condition in which the center of pressure is assumed at a location corresponding to a fraction of $C_{L\max}$ and the total load on the wings considerably less than in High Angle of Attack. This loading condition is normally critical for the rear lift truss, some members of both drag trusses, and some fuselage members just to the rear of the connections to the wing.

In both the High and Low Angle of Attack conditions the dominating forces are the lift on the wings and the parallel inertia forces caused by the inability of the weight and tail load to balance that lift. The drag loads and opposing inertia forces must be considered, but are of minor importance. The propeller thrust and tail load are so small that they are arbitrarily assumed to be of whatever magnitudes are necessary to bring the other forces into equilibrium. The weight acts vertically through the centers of gravity of the various elements of the airplane, while the inertia forces which are proportional to the weights also act in the same locations. Although the weight and inertia force of each element are not necessarily parallel, for simplicity of computa-

tions they are assumed to be not only parallel to each other but also parallel to the resultant air pressure on the wings. The labor of computation is greatly reduced by this assumption, which is as reasonable as any that could be made and does not introduce serious errors.

2 : 7. Inverted Flight — Sudden decreases in angle of attack must be considered as well as sudden increases. These may be produced by flying through bumpy air or by intentional manipulation of the controls to cause the airplane to dive. Such decreases in angle of attack may carry the wing past the angle of zero lift and into the range of negative lift in which the net air pressure of the wings is a "down" load. The inertia forces then act "up" and as they may become greater than the weights it is necessary to provide the Inverted Flight condition in which the lift loads and the resultant of weight and inertia forces act in directions opposite to the normal. This condition also takes care of the possibility of the pilot's flying the airplane upside down, either intentionally or due to being thrown into an inverted position during a storm.

Since the center of pressure location for angles of negative lift is approximately the same as for large values of positive C_L, the Inverted Flight condition is practically the High Angle of Attack condition with the directions of the forces reversed. Since the maximum negative value of C_L is much smaller than the maximum positive value, the accelerations and total forces involved in the maneuvers represented by the Inverted Flight condition are much smaller than in High Angle of Attack. Usually an analysis for the Inverted Flight condition is made for the wings and cabane only, though it may have to be carried into the fuselage. It is usually critical for the front lower wing spar, the landing wires, some members of the cabane, and the cross tubes in the fuselage to which the cabane is attached.

2 : 8. Landing Conditions — The range of landing attitudes that must be considered by the airplane designer is that between a landing in which the wheels and tail skid touch the ground simultaneously, a "three-point landing," and a "two-wheel" or "level landing" in which the resultant ground reaction passes through the center of gravity while the thrust line is horizontal. If the tail-skid should hit the ground before the wheels, the impact would produce a ground reaction that would cause the airplane to rotate and make the wheels touch very quickly, so that the attitude of the three-point landing would be reached before the ground reactions had reached their full magnitudes. If, in a two-wheel landing, the resultant ground reaction should pass behind the center of gravity it would produce a couple that would cause the airplane to turn over on its nose. The first of these limiting

attitudes of landing is the basis of the Three-Point Landing condition, and the second that of the Level Landing condition.

At the instant the wheels touch the ground the velocity of the airplane has a downward vertical component, the quick elimination of which indicates the existence for a short time of an upward acceleration, and downward inertia forces. This upward acceleration is produced by the ground reactions representing the resistance of the ground to further downward movement of the airplane. The magnitude of the ground reactions depends primarily on the weight of the airplane, the magnitude of the vertical component of its velocity when it alights, and the effectiveness of the shock-absorbing mechanism in prolonging the period of upward acceleration and making that acceleration uniform. The lift forces on the wings and tail assist the ground reactions, but their effect is relatively small and is neglected. There is also friction between the landing gear and the ground that tends to slow down the airplane, i.e. produce a backward acceleration, and the accompanying forward inertia forces. The drag also tends to slow down the airplane while the propeller thrust may tend to increase its speed, but these forces are either neglected or are assumed to be taken care of in the friction between the landing gear and the ground.

In practice the same total vertical loads are assumed to act in both the Three-Point Landing and the Level Landing conditions. By doing this and neglecting all horizontal forces in the former, but assuming conservative values for them in the latter condition, all of the intermediate attitudes of landing are covered.

The two conditions discussed above fail, however, to cover two important possibilities, and two additional conditions are provided for them. The ground reaction may have a sidewise component due to landing with the wings not parallel to the ground or in a side slip. Transverse ground reactions and inertia forces are therefore combined with the other forces of the Level Landing condition to define the condition of Level Landing with Side Load. No Three-Point Landing with Side Load condition is specified as the one side load condition has been found sufficient. Another possibility is that the functioning of brakes will produce local stresses greater than are developed in the same type of landing when the brakes are not used and this gives rise to the necessity of investigating a Braked Landing condition.

The landing conditions are usually critical for all members of the landing gear, most of the upper longeron, many of the web members of the fuselage side trusses, and the cross tubes in the fuselage to which the landing gear struts are attached.

The landing conditions for which the structure is analyzed are not

the only ones that are encountered in practice, but represent the extreme locations for the direction of the ground reaction consistent with satisfactory landing. Most actual landings are made with the direction of the resultant ground reaction somewhere between those assumed for the Level and Three-Point Landing conditions. The designer should therefore avoid the temptation to design a structure with only the specified conditions in mind, but should be sure that he adopts one that will function properly when the ground reaction is in any intermediate location.

2 : 9. Minor Loading Conditions — None of the major loading conditions discussed above requires loads on the control surfaces large enough to ensure adequate strength and stiffness either in the surfaces or the members connecting them to the central portion of the structure. To remedy this defect, three loading conditions are used: Maximum Aileron Load, Maximum Stabilizer and Elevator Load, and Maximum Fin and Rudder Load. All of these are empirical loads based upon experience. They are assumed to be imposed on the control surfaces in question, and the stresses caused by them are followed until parts of the structure are reached for which they are clearly not critical. The Maximum Aileron Load is seldom critical for anything except the ailerons and such structure as is needed to support them from the rear spar. The Maximum Stabilizer and Elevator Load is not only critical for those surfaces but also for the rear part of the lower longeron and some of the web members of the fuselage side trusses. In fact it is critical for so many important members that it becomes, in effect, a main loading condition for the fuselage. The Maximum Fin and Rudder Load is not only critical for these surfaces but also for the web members of the rear portion of the top and bottom fuselage trusses.

The Leading Edge loadings are used to ensure adequate strength in the leading edge portions of the wings and tail surfaces to withstand the heavy local concentrations of air pressure found in those locations in pressure distribution tests.

The condition of a side load on the engine mount is provided to ensure adequate strength in that part of the structure against the transverse inertia loads produced during turns and also any sidewise vibrations that may exist.

The Nosing Over condition covers the possibility that the airplane will turn over on its nose in a bad landing. It is not intended to make necessary a structure so strong that it would not be injured by such an accident, but rather to provide one that might absorb enough of the impact that the crew and passengers of the airplane would escape serious injury.

In many types of chassis the loads produced in all the members in a one-wheel landing would be just twice those in a two-wheel landing. In others, the loads on the two wheels tend to neutralize each other in some members and a one-wheel landing would be more than twice as severe on those members as one of the two-wheel type. The One-Wheel Landing condition is provided to obtain a proper design for such members when they exist.

In the main wing loading conditions it is assumed that the loads on the wing are symmetrical. That is not correct when the airplane is going into a bank and certain other maneuvers. Although the total loads in these maneuvers are not so large that they must be considered in the analysis of the wings themselves, a small lack of symmetry will often produce critical loads in the cabane, and the Unsymmetrical Cabane Load condition is provided for this condition.

One feature that all of the minor loading conditions have in common is that the rules by which they are applied are highly arbitrary, none of them resting on as sound a theoretical basis as the main loading conditions. Research is in progress, particularly in the form of full-scale flight tests to determine accelerations and pressure distributions, that should throw much light on most of the minor loading conditions. As the pertinent data are obtained and studied, the rules for these conditions will undoubtedly be modified to make them more closely represent actual loadings.

2 : 10. Wing and Tail Flutter — One of the things which a designer must bear in mind but for which there are no definite rules is that his structure must be as rigid as possible. Flexible surfaces are liable to oscillate due to the vibration of the engine or to periodic changes in the flow of the air over them, and if the period of the forces involved coincides with the natural period of vibration of the surface an unstable oscillation or "flutter" may develop. The amplitude of such an oscillation increases rapidly and, as it induces stresses in the structure which increase about in proportion to the amplitude, it may cause failure of the structure itself. Such failures have occurred in several cases on high speed monoplanes and one is known to have occurred on the upper wing of a biplane which was unsupported for a considerable span.

While no definite data have been obtained on the conditions favoring flutter, and no rules or definite criteria have been established, it appears that wing flutter at least is produced by unstable air flow near the wing tips. As a consequence of such flow oscillations may be set up in the ailerons and if the aileron control system permits, the amplitude of these oscillations may build up until they become dangerous as

regards the aileron or they may be transmitted into the wing and cause the whole wing structure to flutter. An analysis of the forces developed under such conditions indicates that a structure which is torsionally stiff will have a greater tendency to resist flutter than one which is flexible, hence wing structures involving long cantilevered portions should be as stiff as possible torsionally. Such stiffness may be obtained by using plywood covering on the wing or by using a double system of internal drag trussing that is greatly over-strength and the elements of which are spaced as far apart as possible in the given wing structure. A rigid control actuating mechanism, such as that provided by a series of push rods, will reduce the tendency of unstable oscillations of the ailerons and will reduce the possibility of inducing flutter in the wing itself.

Similar methods may be used to prevent or reduce tail vibrations, although the simplest way to eliminate flutter on such surfaces — and one which adds relatively little to the weight or resistance of the airplane — is to brace the structure by a tube which is rigidly connected to the fuselage. Making the elevator torque tube somewhat over-size and having it continuous from the tip of one elevator to the tip of the other is also advantageous as it prevents oscillation of one elevator surface with respect to the other.

Studies of wing flutter are being made in several laboratories and it is anticipated that definite criteria may soon be established to determine when it is liable to occur. The present indications of such studies, indications which have been corroborated by actual airplanes, are that the best preventive of flutter is stiffness as regards both torsion and flexure.

2 : 11. Loading Specifications — In the preceding paragraphs the various loading conditions have been discussed in general terms, nothing having been said about the details. This was done because, while the basic loading conditions are about the same in all countries, and remain the same in their essentials, the detailed rules for computing external loads differ from place to place and are constantly being changed. In this country the rules that must be followed in order to obtain an airworthiness certificate for a commercial airplane are published by the Department of Commerce, and before starting on a new model a designer should be sure to obtain a copy of the latest issue. Similar requirements are promulgated by the Army and Navy for service types of airplanes.

Though it is not intended to comment on the details of the rules a few remarks should be made about the form in which loads and strengths are expressed in the Department of Commerce rules, as this differs from

the system used in most other lines of structural engineering. The "design loads" for an airplane or its parts are what might be termed the "minimum allowable ultimate loads," i.e. the smallest loads that may cause failure of the structure without its being considered too weak. In the major loading conditions, such as High and Low Angle of Attack, Level and Three-Point Landing, the design loads are specified in terms of "load factors." That means that the weights of the airplane and its parts are to be multiplied by the required load factor and the air pressures, ground reactions, etc., are to be of such magnitudes that the entire system of forces will be in equilibrium. The load factors specified are made large enough to include an allowance for the inertia forces and also a "factor of safety," the latter being the ratio of the smallest load that should cause failure to the maximum probable load. As the weight and inertia forces are assumed to act in the same direction, the factor of safety is always less than the load factor. In a few cases, notably the control surface loading conditions, the design loads are specified as so many pounds per square foot on the surfaces in question, and here too the load specified is the one under which the structure may be on the point of failure. In other words, the design load, whether specified in terms of load factors or pounds per square foot, is an ultimate load.

To determine whether a member is satisfactory, the normal procedure is to compute the load that would cause it to fail and compare that to the design load. In all but the simplest cases it is most convenient to compute the internal stresses caused by the design load and compare them with the maximum internal stresses of the same character that the member can stand without failure, rather than to compare external loads. The quantity

$$\frac{\text{allowable load}}{\text{design load}} - 1$$

or

$$\frac{\text{allowable unit stress}}{\text{unit stress due to design load}} - 1$$

is known as the "margin of safety." If the allowable load or unit stress is the larger, the margin of safety will be positive and the member satisfactory. If the allowable load or strength is less than the design load, the member is obviously unsatisfactory, and this is indicated by the margin of safety being negative.

PROBLEMS

2 : 1. Draw a figure similar to Fig. 2 : 2 showing the forces acting when the airplane is flying at a high angle of attack of the wings but the line of flight is sloping downward, and write the corresponding equations of equilibrium.

2 : 2. Draw a similar figure for the case of entry into a dive, assuming the line of flight to slope downward and the angle of attack to be in the range of negative lift, and write the corresponding equations of equilibrium.

2 : 3. An airplane the stalling speed of which is 50 m.p.h. is flying at 120 m.p.h. What is the maximum lift load that can be imposed on the wings by an abrupt zoom? By flying into bumpy air?

2 : 4. An airplane weighs 3000 lb. fully loaded, and the specified load factor in the Three-Point Landing condition is 6.0. What will be the total of the weight and inertia forces for which the airplane must be designed in this loading condition?

2 : 5. The design load for a certain lift wire is 2000 lb. It is proposed to use for that member a wire that will carry just 1900 lb. without failure. What is the margin of safety for that wire? Would it be satisfactory?

CHAPTER III

REACTIONS, SHEARS, MOMENTS, AND INFLUENCE LINES

Once the external loads on an airplane have been ascertained, it is necessary to determine their effect on the airplane structure. To do this, the structure must first be resolved into its elements, which take the form of beams, trusses, columns, etc., and the loads acting on each computed. Then each element can be considered by itself, the internal stresses caused by the design load computed and compared to the corresponding allowable stresses, and the margin of safety determined. The methods employed for doing this will form the subject matter of most of the remainder of this volume. Since these methods are applicable to many types of structures other than airplanes, such as bridges, buildings, machines, etc., they will be treated as general methods of analysis. As the reader is interested primarily in airplanes, however, most of the illustrative examples and problems will be taken from the aeronautical field.

3 : 1. Fundamental Principles — All of the methods that are commonly used in the analysis of structures are based on the assumption that the external or outer forces acting on the whole structure or any of its parts, no matter how small, are in equilibrium. This assumption is correct when the structure, or the part of it being considered, is at rest or in unaccelerated motion. If it is in accelerated motion and inertia forces are added to the system in accordance with d'Alembert's principle discussed in Art. 2 : 2, the assumption is still justified.

Mathematically the assumption of equilibrium is applied by writing algebraic equations representing the relations that must exist between forces which are in equilibrium. By writing such equations for the external forces acting on the whole structure we obtain relations between the known forces, the loads, and the unknown external forces, the reactions. By writing them for the external forces on a part of the structure and the internal forces normally produced on that part by the remainder, relations are obtained between some of the external and some of the internal forces. If the part is one on which no external forces act, the relations obtained will be between internal forces exclusively. It should be noted that these "internal" forces are internal only with respect to the whole structure. So far as the part on which they act is concerned, they may be considered as external forces.

This method of analyzing the forces acting on or in a structure is often called the "free body method," since the whole or a part of the structure is assumed to act as a free body subjected to a system of external forces. That part of the structure which is considered the free body is assumed to be isolated from the remainder, which is removed and replaced by external forces on the free body identical with those previously exerted on it by the part assumed to have been removed. The section cutting off the free body from the remainder of the structure may be a plane surface, or may have any other shape, provided that it completely severs the free body from the remainder of the structure. In the application of the method, care must be taken that the external forces assumed to act on the isolated portion represent all the forces that were acting on it from the remainder of the structure through the boundary section.

If the entire structure is taken as the free body and if in the application of the assumption of equilibrium we can write as many independent equations as there are unknown external forces, those unknown forces can be computed by solving the equations, and the structure is said to be statically determinate with respect to the outer forces. If the assumption provides more independent equations than there are unknown forces, the structure is, in general, unstable and would not stand up regardless of the strength of its individual members. If there are more unknown outer forces than there are independent equations that can be obtained from the assumption of equilibrium, the structure is said to be redundant or statically indeterminate with respect to the outer forces.

Most structures, certainly all airplanes, are aggregations of struts, tie rods, beams, and similar parts which may be called design elements. By choosing suitable boundaries for the free bodies it is possible to write equations expressing the relations between the external forces on the whole structure and all the external forces on the various elements. If as many independent equations can be written in this manner as there are unknown forces, the structure is said to be statically determinate with respect to the inner forces.

When statically indeterminate structures are analyzed, whether they are indeterminate with respect to the outer forces or to the inner forces, it is necessary to resort to the principle of consistent deformations in order to obtain as many independent equations as there are unknown forces in the system being investigated. This principle and its application will be discussed in more detail in Chapter XIV, and it is sufficient here to remark that in order to apply it, it is necessary to know the sizes and cross-sections of the members of the structure, whereas

in the case of a statically determinate structure it is sufficient to know the applied external loads and the locations of the center-lines of the design elements. The next few chapters will therefore be devoted to the methods of determining the external forces on the individual elements, and the sizes of members of the different kinds needed to carry those forces.

3 : 2. Laws of Statics — Statics may be defined as the science that treats of systems of forces which are in equilibrium. These forces may act in space in any direction or they may all act in one plane.

Since it is generally less difficult to handle a system that lies in one plane than to deal with forces acting in space it is customary, wherever possible, to resolve a three-dimensional structure into a number of planar structures. An airplane is particularly susceptible to such resolution as it is, in the case of conventional types, composed of a box-like body or fuselage, made up of the two side trusses, the top and bottom trusses each of which lies essentially in a plane, and a series of planar bulkheads. The wing cellule is composed of planar wing panels and interplane struts which are used in combination with various wires or struts as parts of a planar "lift," "landing" or "drag" truss. The problem of analysis thus becomes one of resolving the applied forces into the planes of the various combinations of members and applying the equations of statics to determine the forces developed and the stresses produced in the members themselves.

A planar structure having the applied loads acting in the plane of its members is in equilibrium if the following conditions are fulfilled:

1. The algebraic sum of the components of all the forces acting parallel to any axis must equal zero.
2. The algebraic sum of the moments of all the forces taken about any axis normal to the plane of the forces must equal zero.

In practice it is generally convenient to resolve all forces into components parallel to two rectangular axes in the plane of the forces. If these axes are called X and Y, and ΣX be used to represent the sum of the components parallel to the X axis, ΣY the sum of those parallel to the Y axis, we can write two of the equations of equilibrium: $\Sigma X = 0$ and $\Sigma Y = 0$. If, as is customary, the axes are chosen in the vertical and horizontal directions, then $\Sigma V = 0$ and $\Sigma H = 0$.

Using ΣM to designate the sum of the moments of all of the forces about any axis at right angles to their plane, then, from the second condition, we obtain the third equation of equilibrium, $\Sigma M = 0$.

When a structure is such that the forces acting in its various members can be computed by suitable applications of the above equations of equilibrium it is "statically determinate." Structures that cannot be

analyzed completely by applications of the above equations are "statically indeterminate" and require the use of one of the methods of consistent deformations for their complete solution. A structure may be statically determinate, or indeterminate, with regard to the outer forces, with regard to the inner forces, or with regard to both outer and inner forces.

3 : 3. Reactions — The first step in the analysis of a structure, once the magnitude and distribution of the external loads imposed upon it have been satisfactorily determined, is the computation of the reactions, the external forces which hold the applied loads in equilibrium and thus complete the system of outer forces on the structure. In general there are two reactions on any statically determinate planar structure so that it is necessary to deal with two unknown forces, each of which has three characteristics: magnitude, direction and point of application. There are therefore six properties of the reactions to be ascertained in order that the reactions may be completely determined, yet to do this there are available only the three equations of equilibrium.

Three of the properties may sometimes be predetermined by arranging the details of the fittings or connections in such a way that the direction of one and the points of application of both of the reactions will be fixed. This is readily done in bridge construction by fixing one end of a span to a pier or abutment by bolts and resting the other end on rollers. Since the bolted end will transmit both horizontal and vertical forces the reaction is fixed as to point of application, but not as to magnitude or direction. The reaction at the end resting on rollers must act normal to the supporting surface so it is fixed both as to direction and point of application, but not as to magnitude. The unknowns to be determined by the equations of equilibrium are thus reduced to three. In airplane structural design it is seldom possible to fix the direction of either reaction by the use of rollers or slotted holes. The direction is, however, established when a wire is used to furnish a reaction since the force must act along the axis of the wire.

Where it is not possible to prearrange the characteristics of the reactions, satisfactory results may often be had by assuming the magnitude of one of the reactions or, more commonly, of a component of one of them. This can generally be done without jeopardizing the safety of the structure as a whole though it requires that special attention must be paid to the portions near the reactions and that the fittings and other details must be proportioned to withstand the stresses obtained with the most conservative assumptions. Where an exact determination is required, one of the methods based on the deformation of the structure, such as the method of least work, must be used.

If there are but three unknown properties of the reactions, a structure is statically determinate as regards the outer forces; if there are more than three it is statically undetermined unless some special conditions are encountered from which other equations may be obtained. If there are fewer than three unknowns the structure is generally unstable and will tend to move bodily with respect to the supports unless the applied loads are so arranged as to fulfill certain special conditions.

Figures 3 : 1 and 3 : 2 illustrate some of the conditions existing with different types of support, it being assumed that the loads and all dimensions are known in each case. In Fig. 3 : 1 the point of application of the left reaction, the components of which are shown, is fixed by the pin connection of the wing to the fuselage but both the magnitude and direction are unknown. The point of application and direction of R_R are fixed by the point of attachment and slope of the wire which provides the support. Only the magnitude is unknown. There are then but three unknown characteristics for R_L and R_R and they are determinable from the three equations of equilibrium. Hence the structure is statically determinate as regards the outer forces.

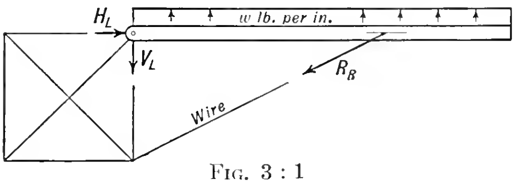


Fig. 3 : 1

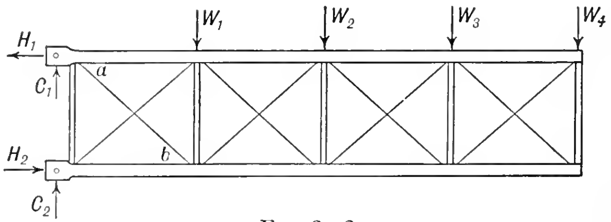


Fig. 3 : 2

Fig. 3 : 2 represents a plan view of an internal drag truss in a wing, the structure being attached to the fuselage or cabane struts at 1 and 2. The points of application of the two reactions are fixed but not their magnitudes nor directions. There are, then, four unknowns so the structure is statically indeterminate as regards the outer forces. Since this is a condition often occurring on an airplane it is customary to make certain assumptions regarding the reactions so that they may be computed by the equations of equilibrium. In this case it would be reasonable to assume that C_1, the component of R_1 parallel to the wing chord, was equal to $W_1 + W_2 + W_3 + W_4$ since the wire *ab* would

act to carry the chord component of the loads to that point. If such an assumption is made it follows that R_2 is horizontal since there are no other chord loads on the structure. Having fixed the magnitude of the chord component of R_1 the number of unknowns has been reduced to three and the structure becomes statically determinate. Another reasonable assumption is that the chord components C_1 and C_2 of R_1 and R_2 are equal. This might be made without affecting the forces in the members materially, but it is obvious that the loads to be carried by the inner compression strut and by the fittings connecting the front and rear spars to the fuselage would be considerably altered. It would therefore be necessary to investigate the forces carried by the compression strut and fittings under two or three, possibly more, assumptions as to the division of load between R_1 and R_2 so that the structure could be designed for the most conservative conditions.

3 : 4. Computation of Reactions — In determining the magnitude of the reactions on a structure it is generally less difficult to compute the components of each reaction than to determine the actual magnitude of the reaction itself. Since the reactions for one member in a structure are often utilized as "loads" for another it is often desirable to obtain the reactions in terms of the components parallel with and at right angles to the supporting member. In this way the components producing direct tension or compression in the supporting members are separated from those that produce bending, thus facilitating the analysis of the supporting members when the time comes to investigate them. In many cases it is more desirable to obtain the components parallel with and perpendicular to the member supported.

The following procedure is recommended for the use of the student and it is suggested that he follow it exactly until he has mastered it.

1. Draw a careful sketch of the structure and indicate the components, preferably vertical and horizontal, of the outer forces. This sketch need not be exactly to scale but it is desirable that it should be. In any case it should not be materially distorted.

2. Indicate on the sketch by arrows and letters the components of the loads and reactions. The use of H and V to designate the horizontal and vertical components with suitable subscripts such as R and L for the right and left reactions, 1, 2, 3, etc., for the loads is recommended. The directions assumed for the reaction components are immaterial so long as the direction assumed is used consistently throughout the computations.

3. Ascertain whether or not the structure is statically determinate with respect to the outer forces.

4. Determine the unknown H and V components by the applica-

tion of the equations of statics, i.e., $\Sigma H = 0$, $\Sigma V = 0$ and $\Sigma M = 0$. In order to keep the signs of the terms in the equations correct it is recommended that loads acting upward or to the right be considered positive and that clockwise moments be positive. If this is done, a positive result shows that the component in question acts in the direction assumed, but not necessarily upward or to the right.

When the magnitude of each of the components has been obtained, the magnitude of the reaction itself may be found by taking the square root of the sum of the squares of the components. The direction may be determined by the direction and magnitude of the components.

It is good practice for the beginner to compute each reaction by taking moments about the other and then check them by determining whether or not ΣH and ΣV for loads and reactions are zero. If this is done it will often be found that one of the forces has been omitted entirely or has been used with an incorrect sign. It is impossible to exercise too great care with the signs used for the forces acting on an airplane structure since some — notably the lift on the wings — act upward in the design conditions, while others — such as the weights of the pilot, passengers, gasoline, etc. — act downward, and still others may act in either direction. In the following illustrative problems the loads are assumed to act in the directions ordinarily employed in design and the student is advised to give especial attention to the way the signs are used with the various terms.

3 : 5. Illustrative Problems — Reactions — 1. Determine the horizontal and vertical components of the reactions on the wing spar shown in Fig. 3 : 3.

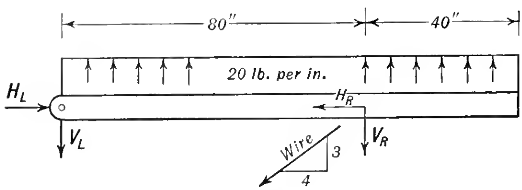


Fig. 3 : 3

Solution. — First, is the structure statically determinate as regards the outer forces? There are two unknowns for the left reaction, the magnitude and direction, and one, the magnitude, for the right. The direction of the right reaction is fixed by the slope of the wire. We have, therefore, three unknowns so the structure is determinate.

Solving for V_R by taking moments about the pin joint at the left end of the spar, $\Sigma M = 0$.

$$\text{Then, } -20 \times \frac{120^2}{2} + 80\, V_R - 0 \times H_R + 0 \times H_L - 0 \times V_L = 0$$

$$V_R = 1800 \text{ lb.}$$

Solving for V_L by taking moments about the right reaction:

$$-80 \times V_L + 20 \times \frac{80^2}{2} - 20 \times \frac{40^2}{2} + 0 \times H_L - 0 \times H_R - 0 \times V_R = 0$$

$$V_L = 600 \text{ lb.}$$

From $\Sigma V = 0$,

$$20 \times 120 - V_L - V_R = 2400 - 1800 - 600 = 0.$$

The load in the wire is $\frac{5}{3} \times 1800 = 3000$ lb. $= R_R$.
Its horizontal component is $\frac{4}{3} \times 1800 = 2400$ lb.
From $\Sigma H = 0$: —

$$H_L - H_R = 0 = H_L - 2400$$

$$H_L = 2400 \text{ lb.}$$

$$R_L = \sqrt{V_L^2 + H_L^2} = 2475 \text{ lb.} = R_L.$$

2. Determine the horizontal and vertical components of the reactions for the hangar roof truss shown in Fig. 3 : 4 assuming a wind pressure of 20 lb. per sq. ft. normal to the surface of the roof. Assume the trusses to be 25 ft. apart so that an intermediate truss will carry a section of roof 25 ft. long.

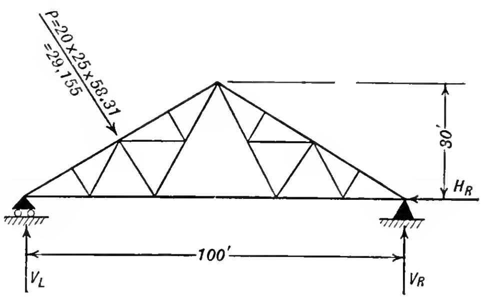


FIG. 3 : 4

Solution. — This structure is statically determinate as regards the outer forces since there are but three unknowns: the magnitude of the left reaction, since it is on rollers, and the magnitude and direction of the right. Taking moments about the left reaction to determine the right and using the resultant of the wind load, acting as shown in the figure,

$$\Sigma M = 0 = 29{,}155 \times \frac{58.31}{2} - 100 \times V_R - 0 \times H_R + 0 \times V_L,$$

$$V_R = 8{,}500 \text{ lb.}$$

$V_L + V_R$ must equal the vertical component of P since $\Sigma V = 0$,

$$V_L + V_R - \frac{50}{58.31} \times 29{,}155 = 0$$

$$V_L = 16{,}500 \text{ lb.}$$

H_R must equal the horizontal component of P since $\Sigma H = 0$ and H_L is 0 by the arrangement of the supports.

$$H_R = \frac{30}{58.31} \times 29{,}155 = 15{,}000 \text{ lb.}$$

A check will now be made by taking moments about the right reaction, using the components of P instead of the resultant.

$$\begin{aligned}\Sigma M = 0 &= 100\ V_L - 75\ V_P + 15\ H_P \\ &= 100 \times 16{,}500 - 75 \times 25{,}000 + 15 \times 15{,}000 = 0.\end{aligned}$$

3 : 6. Shear and Moment — The shear on any section of a beam or truss is the resultant of the outer forces on either side of the section tending to produce slipping along the section. It is equal to the algebraic sum of the components parallel to the section of all the forces, including the reactions, on either side of the section.

The moment at any section through a beam, or truss, is equal to the algebraic sum of the moments of the forces on either side of the section taken about an axis normal to the plane of the forces and passing through some point in the given section. In a beam this moment is commonly taken about an axis through the centroid of the beam cross-section and it is called the "bending moment." In a truss the moment is taken about an axis through the center of moments for the member in question.

3 : 7. Conventions for Signs of Shears and Moments — In airplane structures forces acting vertically upward or horizontally toward the right are considered positive. When the resultant of the forces to the left of a section is positive the shear is positive and when the character of the resultant moment on any section is such as to cause compression in the upper fiber of a beam, or upper chord of a truss, it is positive.

Great care must be exercised in using the proper signs, or the results will be misleading. This is of especial importance regarding the moments, since it is imperative in the design of a beam or truss to know which part is to carry compression and which tension.

3 : 8. Curves of Shear and Bending Moment — The magnitude of the shear in a body at any given section, due to a set of co-planar forces, may be obtained by resolving each force, including the reactions, into components parallel and normal to the section. The algebraic sum of the components parallel to the section of all the forces on either side of the section is the shear.

If the magnitude of the shear is determined for several sections along a beam or truss, a curve may be constructed such that at any section the ordinate to the curve is equal to the shear. Such a curve is called a curve of shears. A similar curve can be drawn to give the bending moment at any section in a beam or to give the moment about a series of points such as the panel points of a truss. Such a curve is called a curve of moments or a moment diagram.

In drawing these curves it is customary to plot positive shears and

moments above the axis, negative, below. The shape of the curves will vary with the loads used and the student should become familiar with the types of loading which give a continuous smooth curve, a series of smooth curves, or a series of straight lines. Such a knowledge will enable him to compute the values at all critical points on the curves and fair in the others so that they show the trend of the change in shear or moment.

The following diagrams show curves of shear and bending moment for a simple case, the weight of the beam itself being neglected.

3 : 9. Relations between Shear and Moment — The most important relation existing between shear and moment is the mathematical rela-

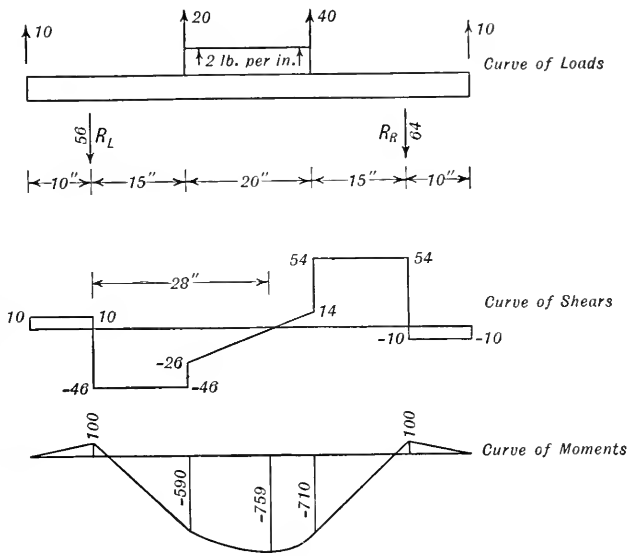


FIG. 3 : 5

tion between the equation for the curve of shears and that for the curve of moments. Figure 3 : 5 shows curves of shear and moment in a beam and, for the loading given, it is obvious that neither set of curves is expressible mathematically as a continuous function. Each is made up of five segments, that from the left end to the left reaction, that from the reaction to the 20-lb. load, that under the uniform load, that between the 40-lb. load and the right reaction and that between the reaction and the right end of the beam. Throughout the length of each segment the curves of shears and moments are straight lines or smooth curves which may be represented by simple equations. Considering the segment between the left reaction and the 20-lb. load and

taking the origin at the left reaction the equation for shear throughout the segment is $S = -46$, and for moment, $M = 100 - \frac{690}{15}x = 100 - 46x$. A short study of these equations indicates that the one representing the curve of shears is the derivative, with respect to x, of the equation for the curve of moments, or, conversely, that the moment curve represents the integral of the shear curve, the term 100 representing the constant of integration in this case. More rigorous proofs of this relation may be found in texts on Applied Mechanics or Strength of Materials. It exists in all cases and, being of considerable utility, should be kept in mind.

As a corollary of the relation expressed by $\frac{dM}{dx} = S$ it follows that where S is zero the moment must be a maximum, or minimum, since the mathematical procedure for obtaining maxima and minima is to differentiate and equate the first derivative to zero. Each point where the shear curve crosses the axis is therefore of importance since it locates a point on the moment curve where the slope changes sign. It should be carefully located so that the value of the moment at the section may be determined accurately.

Since the equation for the curve of moments is the integral of that for the curve of shears, plus the constant of integration, the moment at any section on a beam subjected only to loads which are normal to the axis is equal to the algebraic sum of the areas under the shear curve to either side of the section multiplied by the scales to which the shear curve is constructed. If the loads have components parallel to the axis of the beam which develop axial loads in the beam and produce a bending moment at any section or if an external moment is applied at either end of the beam this relation does not hold, since the area under the shear curve does not suffice for the complete determination of the constant of integration.

Another important relation makes it possible to compute the moment at any section of a beam, M_x, when the moment, M_1, and shear, S_1, at any other section, and the loads acting between the sections are known. Expressed algebraically, we have $M_x = M_1 \pm S_1x \pm \Sigma Fa$, where M_1 and S_1 are the moment and shear at section 1; x, the distance between sections 1 and x; and ΣFa represents the sum of the moments about section x of each of the loads acting between the sections. It is difficult to establish the signs of the terms in this equation so that they will fit all loading conditions, but it is a simple matter to determine them in any given case. M_1 will have the sign corresponding with its characteristic at section 1. If the shear at section 1 is of such character

that the moment S_1x tends to increase the compression in the upper fiber of the beam at section x, the term is positive. Similarly the moments, Fa, due to those loads between the sections which tend to produce moments about section x causing compression in the upper fiber at the section are positive, the others are negative. The use of this relation is illustrated below in the computation of the maximum moment and of the moment at the midpoint of the beam in Fig. 3 : 6.

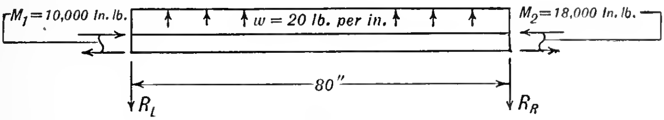


FIG. 3 : 6

The value of R_L obtained by taking moments about R_R is,

$$-80\ R_L + 20 \times \frac{80^2}{2} + 10{,}000 - 18{,}000 = 0$$

$$R_L = 700 \text{ lb.}$$

The value of R_R is, by taking moments about R_L,

$$80\ R_R - 20 \times \frac{80^2}{2} + 10{,}000 - 18{,}000 = 0$$

$$R_R = 900 \text{ lb.}$$

The moment at mid-span is then,

$$M_1 \pm S_1x \pm \Sigma Fa = 10{,}000 - 700 \times 40 + 20 \times \frac{40^2}{2} = -2000 \text{ in.-lb.}$$

At the section of zero shear, which occurs at $x = \dfrac{700}{20} = 35$ inches from R_L, the moment is, $10{,}000 - 700 \times 35 + 20 \times \dfrac{35^2}{2} = -2250$ in.-lb. This is the maximum negative moment in the span. It is to be noted that the positive moments at the ends are greater in this case, and would be critical in the design of a beam to carry this system of loads.

This relation can be used to great advantage in the computation of the bending moment at several sections on a beam, being particularly suitable for the case of a cantilever subjected to uniformly distributed or varying loads. Beginning at a section where the moment is known, in this case at the end where it is zero, it is possible to determine the bending moment at as many sections along the span as may be necessary and the computations may be tabulated as shown in the following example to reduce the labor to a minimum.

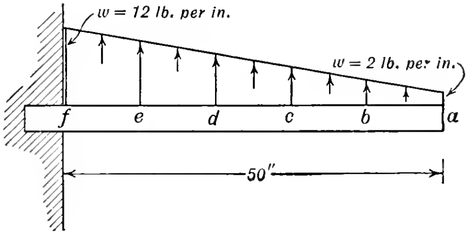


FIG. 3 : 7

TABLE 3 : 1

Section	Load per In. w	Average Load	Dist. between Sects x	Load between Sects F	Arm to Centroid a	Moment $M'=Fa$	Shear $S=\Sigma F$	Dist. between Sects x	Moment $M''=Sx$	Moment at Section
a	2						0			**0**
		3	10	30	4.44	133		10	0	
b	4						30			133
		5	10	50	4.67	233		10	300	
c	6						80			666
		7	10	70	4.76	333		10	800	
d	8						150			1800
		9	10	90	4.81	433		10	1500	
e	10						240			3733
		11	10	110	4.85	533		10	2400	
f	12						350			6666

That the above result is correct may be shown by taking moments about section f, whence

$$M_f = 2 \times \frac{50^2}{2} + \frac{12-2}{2} \times 50 \times \frac{50}{3} = 2500 + 4166 = 6666$$

The distances, a, the arms to the centroids of the areas under the loading curve between any two sections, were taken from Table 3 : 2. It is to be noted that the assumption that the centroid of each area lies midway between the sections bounding the area would result in slightly higher values for M', thus giving conservative values for the moments at the various sections. This assumption is customarily made in practice unless the values are desired with an unusually high degree of precision.

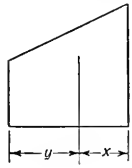
The other columns in the table, with the exception of the last, are self-explanatory and need no further discussion. The values for any section given in the last column are obtained by taking the value of the moment at the preceding section plus the shear at that section

times the distance between the sections plus the moment due to the load between the sections, whence

$$M_d = M_c + S_c x_{cd} + F_{cd} a_{cd} = 666 + 800 + 333 = 1800$$

TABLE 3 : 2

DISTANCES FROM SIDES OF A TRAPEZOID TO ITS CENTROID



Ratio	Distance x	Distance y	Ratio	Distance x	Distance y
1.01	0.4992	0.5008	1.70	0.4568	0.5432
1.02	0.4984	0.5016	1.75	0.4545	0.5455
1.03	0.4976	0.5024	1.80	0.4523	0.5477
1.04	0.4968	0.5032	1.85	0.4502	0.5498
1.05	0.4960	0.5040	1.90	0.4482	0.5518
1.06	0.4952	0.5048	1.95	0.4462	0.5538
1.07	0.4944	0.5056	2.00	0.4443	0.5557
1.08	0.4936	0.5064	2.10	0.4409	0.5591
1.09	0.4928	0.5072	2.20	0.4375	0.5625
1.10	0.4920	0.5080	2.30	0.4343	0.5657
1.11	0.4912	0.5088	2.40	0.4312	0.5688
1.12	0.4905	0.5095	2.50	0.4284	0.5716
1.13	0.4898	0.5102	2.60	0.4259	0.5741
1.14	0.4890	0.5110	2.70	0.4233	0.5767
1.15	0.4883	0.5117	2.80	0.4209	0.5791
1.16	0.4877	0.5123	2.90	0.4188	0.5812
1.17	0.4870	0.5130	3.00	0.4168	0.5832
1.18	0.4862	0.5138	3.20	0.4128	0.5872
1.19	0.4855	0.5145	3.40	0.4090	0.5910
1.20	0.4849	0.5151	3.60	0.4060	0.5940
1.22	0.4835	0.5165	3.80	0.4030	0.5970
1.24	0.4822	0.5178	4.00	0.4000	0.6000
1.26	0.4809	0.5191	4.20	0.3975	0.6025
1.28	0.4795	0.5205	4.40	0.3950	0.6050
1.30	0.4782	0.5218	4.60	0.3928	0.6072
1.32	0.4770	0.5230	4.80	0.3908	0.6092
1.34	0.4758	0.5242	5.00	0.3889	0.6111
1.36	0.4746	0.5254	5.50	0.3848	0.6152
1.38	0.4733	0.5267	6.00	0.3810	0.6190
1.40	0.4721	0.5279	6.50	0.3778	0.6222
1.45	0.4693	0.5307	7.00	0.3750	0.6250
1.50	0.4667	0.5333	7.50	0.3725	0.6275
1.55	0.4641	0.5359	8.00	0.3702	0.6298
1.60	0.4616	0.5384	9.00	0.3668	0.6332
1.65	0.4592	0.5408	10.00	0.3636	0.6364

This method of tabulating the moments can, of course, be used with any type of load distribution and with any type of support and it is

often used when the magnitude of the moment is to be determined at a number of sections. As will be seen later, a similar procedure may be used in determining the deflection of beams.

3 : 10. Influence Lines — In the foregoing paragraphs methods have been described for determining the magnitude of the shear or moment at any section on a statically determinate beam carrying a set of fixed loads. These methods become extremely laborious when it is necessary, as it frequently is, to determine for a number of alternative loadings which will produce maximum shear or moment at a given section. Their use requires that the shear or moment at the given section must be computed for the several arrangements of the loads, so that, by a method of trial and error, that which gives the maximum may eventually be determined. A far simpler and much more satisfactory procedure is to draw a curve to show the changes in shear or moment occurring at the given section when a unit load is moved across the structure and from a study of the changes indicated for the unit load to determine the arrangement of the given loads that will give the desired maximum.

A curve of this type, called an influence line, may be defined as follows: An influence line is a curve such that the ordinate at any point shows the value of the function at the section for which the influence line is drawn due to a load of unity acting at the point where the ordinate is measured. Influence lines are most commonly drawn to show the changes in shear or moment in a beam but they may be constructed for any function which varies with the position of the load, such as the axial load in a truss member.

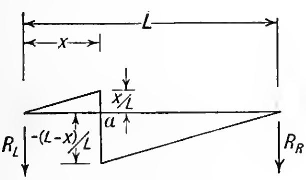
The difference between an influence line for shear and a shear curve is that the ordinate to the influence line shows the shear at a fixed section due to a single unit load applied at the point where the ordinate is measured, whereas the ordinate to the shear curve shows the shear at the section where the ordinate is measured due to all the fixed loads on the structure.

3 : 11. Properties of Influence Lines — By definition, the ordinate to the influence line shows the value of the function at the section for which the influence line is drawn when a load of unity is placed at the point where the ordinate is measured. If, then, a load of any other magnitude is placed at this point the shear, moment or other function at the given section will be the product of the magnitude of the load and the ordinate to the influence line at the point where the load is placed. It follows, therefore, that to obtain the maximum value of a given function due to a single concentrated load that load should be applied at the point where the ordinate to the influence line

is a maximum. For a series of concentrated loads the value of the given function is equal to the algebraic sum of the product of each load and the ordinate to the influence line at the section at which the load is applied, whence the function will have its maximum value when the loads are so disposed that this sum is a maximum.

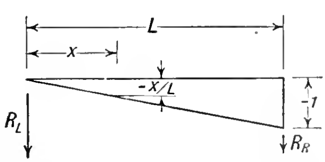
For a uniformly distributed load the value of the given function equals the product of the load per unit length of span times the algebraic sum of the areas between the influence line, the reference axis and the ordinates at the ends of the load. This follows from the fact that if the load over an infinitesimal strip, dL, is $w\,dL$, and the average ordinate to the influence line for the strip is y, the value of the function at the section under consideration will be $yw\,dL$. If, then, we consider loads over all such strips between L_1 and L_2 the value of the function will be the sum of the values of $yw\,dL$, which is represented by $\int_{L_1}^{L_2} yw\,dL$. Where w is constant, as in the case of a uniformly distributed load, this becomes $w\int_{L_1}^{L_2} y\,dL$ or, since $\int_{L_1}^{L_2} y\,dL$ is the area bounded by the influence line, the reference axis and the ordinates at L_1 and L_2 it follows that the value of the function at the given section is w times the area under the influence line between L_1 and L_2. For a uniformly distributed load it is evident that the maximum positive value of the given function will be obtained when the load is distributed over the portion of the structure for which the influence line has positive ordinates; the maximum negative value when the load is applied over the part for which the ordinates are negative. In either case the value of the function will be the product of the load per unit length times the area under the influence line for the loaded portion of the structure.

3 : 12. Examples of Influence Lines for Simple Beams — The unit load is assumed to act vertically upward in the following illustrative cases.



Influence line for shear at section a

FIG. 3 : 8



Influence line for magnitude of right reaction

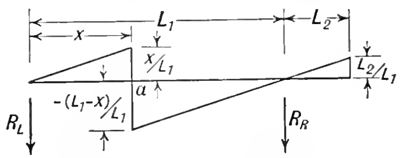
FIG. 3 : 9

If, in Fig. 3 : 8, $L = 50$ in. and $x = 30$ in. then the maximum positive shear at section a for a concentrated load $W = 1000$ lb. would obviously occur when the load was applied an infinitesimal distance to the left

of section a. Its magnitude would be $\frac{30}{50} \times 1000$ lb. $= 600$ lb. If a uniform load, w, of 10 lb. per in. were distributed over the entire span the shear at section a would be

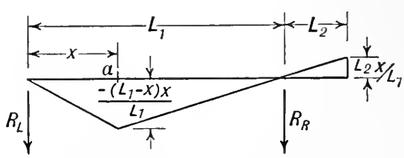
$$S = w\left(\frac{x}{L}\right)\frac{x}{2} - w\left(\frac{L - x}{L}\right)\frac{L - x}{2} = 10\left(\frac{30}{50}\right)\frac{30}{2} - 10\left(\frac{50 - 30}{50}\right)\frac{50 - 30}{2}$$
$$= 90 - 40 = 50 \text{ lb.}$$

For maximum positive shear under this loading only that portion of the beam to the left of section a would be loaded. The shear would then be 90 lb. For maximum negative shear only that portion to the right of a would be loaded and the shear would be -40 lb.



Influence line for shear at section a

FIG. 3 : 10



Influence line for moment at section a

FIG. 3 : 11

3 : 13. Position of Loads for Maximum Shears or Moments — For a simple, end-supported beam it may readily be seen from an inspection of the influence line for shear at any given section that a single concentrated load will cause maximum shear when it is placed an infinitesimal distance to one side or the other of the section. The absolute maximum will occur at the end of the beam when the load acts an infinitesimal distance from the reaction. The moment at any given section will be a maximum when the load is applied at the section and the absolute maximum on the beam will occur when both load and section are at midspan.

In the case of a uniformly distributed load, maximum shear occurs when the beam is loaded over the entire distance between one end of the beam and the given section, the end to be loaded depending on whether maximum positive or maximum negative shear is to be obtained. The shear will have its greatest value when the section is an infinitesimal distance from a support and the load is distributed over the entire span. If the load is distributed uniformly over the entire length of the beam it will produce maximum moment at any given section and the absolute maximum will occur on a section in the center of the span.

When two or more concentrated loads are to be arranged on a structure to produce maximum shear or moment at a given section, the problem becomes somewhat more complex. If it is desired to obtain the arrange-

ment which will cause maximum shear, for example, it is evident from the shape of the influence line that one of the heavy loads must be placed just to the left or just to the right of the given section but when the loads are approximately equal in magnitude it is not always possible to determine by inspection which of the heavy loads should be so located.

Bearing in mind, however, the fact that the shear at the given section is equal to the algebraic sum of all the forces to the left or right of the section, including the reaction, it is possible to determine the change in shear occurring when the loads are moved a short distance by writing an inequality, one side of which indicates the change in the loads that act upward between the given section and the end of the beam, the other side of which indicates the change in the loads that act downward. Assuming that the loads shown in Fig. 3 : 12 are moved from right to left across the span it is obvious from the shape of the influence line that to produce maximum negative shear at a as many of the heavy loads as possible should be applied between section a and the right end of the beam. It may also be seen that if W_2 is moved from a point an infinitesimal distance to the right of a to a point an infinitesimal distance to the left there will be a change in the upward acting loads between R_L and a equal to W_2, but no change in the downward acting left reaction. The negative shear will therefore decrease as the load is moved across the section. If the loads continue to move to the left until W_3 is just to the right of a, the downward load, R_L, is increased but there is no change in the upward loads lying between R_L and a. The negative shear at a is therefore increased as load W_3 is moved up to the section. It is, then, only necessary to determine whether the shear at a is greater with W_2, W_3 or some other load at the section since a maximum will not occur unless one of the loads lies an infinitesimal distance to the right of the section. This may be done by trial by computing the actual shear with each load just to the right of the section or, as mentioned above, by writing an inequality to show the change in the loads that act downward between the left reaction and the section as compared with the change in those that act upward. The latter method is the simpler and will be illustrated by a numerical example.

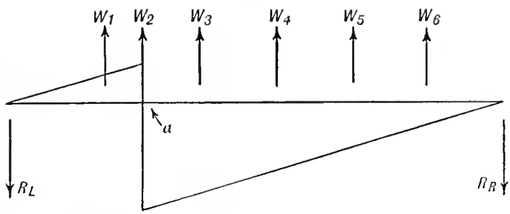


FIG. 3 : 12

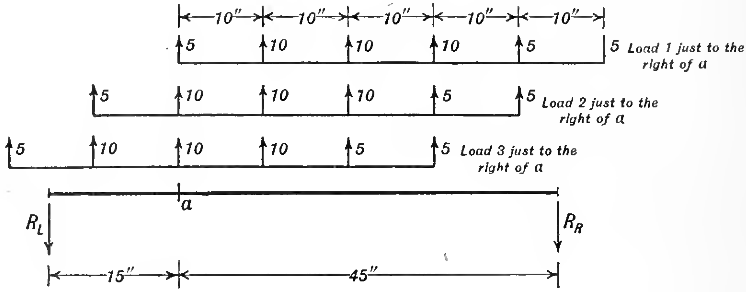


FIG. 3 :-13

Increase in downward loads as loads move from:		Increase in upward loads as loads move from:
First to second position		
$(5 + 10 + 10 + 10 + 5)\frac{10}{60} + 5\left(\frac{5}{60}\right)$	$>$	5
Second to third position		
$(10 + 10 + 10 + 5 + 5)\frac{10}{60} - 5\left(\frac{55}{60}\right)$	$<$	$10 - 5$

The above inequalities show that the shear at a will be increased when the loads are moved from the first to the second position since the change in the downward loads to the left of the section is greater than that in the upward loads between the section and the left reaction. However, when the loads are moved from the second position where load 2 is just to the right of a to the third position where load 3 acts an infinitesimal distance to the right of a the opposite is the case so that the second position of the loads will produce maximum shear at section a.

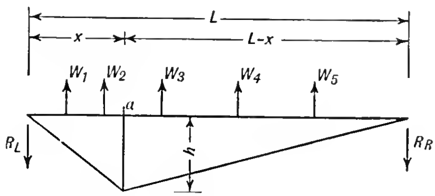
In the first of the above inequalities the first term on the left indicates the change in the downward loads, R_L in this case, due to the loads that were on the span in the first position and that are still on in the second. The second term shows the change due to the 5-lb. load that comes onto the span during the movement of the loads. No loads go off the span, so it is unnecessary to include a term to show the change in R_L due to them. The term on the right side of the inequality shows the change in the upward loads between the left reaction and the section due to the movement of the loads.

The first term on the left side of the second inequality shows the increase in R_L due to the movement of all of the loads that stay on the span. The second term shows the decrease due to the movement of the first load from a point where $\frac{55}{60}$ of it was carried by R_L to a point

off the span where none of it is carried by R_L. On the right of the inequality the first term shows the increase in the loads between R_L and a due to the second load crossing the section as the loads move from the second to the third position. The second term shows the decrease due to the 5-lb. load going off the span. It is obvious that any further movement of the loads to the left will result in a decrease in the negative shear so that, having determined the position of the loads giving maximum shear at section a, it is now possible to compute its value, which is,

$$-\left(\frac{55 + 15 + 5}{60}\right)5 - \left(\frac{45 + 35 + 25}{60}\right)10 + 5 = -28.75 \text{ lb.}$$

The determination of the arrangement that will produce maximum moment at any section on a simple beam may be accomplished in a somewhat simpler manner since the influence line for moment is a triangle lying on one side of the reference axis as shown in Fig. 3 : 14. Since the moment at a under the series of loads shown will be equal to the sum of the loads times their respective ordinates to the influence line the change in moment due to moving the loads d units of length from right to left may be written,



Influence line for moment at section a

FIG. 3 : 14

$$\Delta M = (W_3 + W_4 + W_5)\frac{dh}{L - x} - (W_1 + W_2)\frac{dh}{x}$$

There will be no change in the moment when

$$(W_3 + W_4 + W_5)\frac{dh}{L - x} = (W_1 + W_2)\frac{dh}{x}$$

or when

$$\frac{W_3 + W_4 + W_5}{L - x} = \frac{W_1 + W_2}{x}$$

This is the same as saying that there will be no change in moment when the average load per unit of length to the right of the section is equal to the average load to the left. It is thus apparent that when the average load on the right of the section is greater than that on the left a movement of the loads from right to left increases the moment at the section. When the average load on the left is greater than that on the right a further movement of the loads from right to left causes a decrease in the moment.

The criterion for determining the location of a set of concentrated loads to give maximum moment at any section is thus established. It is: For maximum moment at any given section on a simple, end-supported structure under a given set of concentrated loads, one of the loads must be placed at the section and the others be so disposed that, with the critical load an infinitesimal distance to the right of the section, the average load on the right is greater than that on the left, while with the critical load just to the left of the section the average load on the left is greater than that on the right. This criterion has been established under the assumption that no load came on from the right or went off to the left during the movement of the loads, but it should be obvious that with the small movement involved, the effect of a load coming on or going off the span will be negligible. That it is necessary to have one of the loads lie at the section is apparent since the moment at a given section can be a maximum only when the shear at that section equals or passes through zero, a condition that can only occur under one of the loads.

It should be noted that similar criteria may be developed for any function for which the influence line is made up of two straight lines which lie on one side of the axis so that this "average load method" may be applied to many problems other than the determination of the maximum moment on a simple beam. It should also be noted that the criterion may be satisfied by more than one load so that it may be necessary to compute the actual moment when each of several loads lies at the section to determine which of them produces the absolute maximum.

Using the loads and span shown in Fig. 3 : 13 for illustrative purposes, we have:

Position of Loads	Average Load to Left of *a*		Average Load to Right of *a*	
First load to right	0	$<$	40/45	(No max.)
First load to left	5/15	$<$	35/45	
Second load to right	5/15	$<$	40/45	(Possible)
Second load to left	15/15	$>$	30/45	
Third load to right	10/15	$=$	30/45	(Possible)
Third load to left	20/15	$>$	20/45	

Since the relationship between the average load to the right of the section and that to the left does not change as the first load is moved across the section no maximum occurs with the first load at section *a*.

The second load, however, satisfies the criterion and so may cause the maximum moment. With load 2 at the section the moment is:

$$(55 + 15 + 5)\tfrac{5}{60} \times 15 + (45 + 35 + 25)\tfrac{10}{60} \times 15 - 5 \times 10 = 306 \text{ in.-lb.}$$

With the third load just to the right of the section the average load on the right is equal to that on the left, which indicates that there will be no change in moment at section a when the loads are moved, so long as no load crosses the section or comes on or goes off the span. But, in this case, the average load to the left is greater than that to the right after the load has crossed the section, indicating that the bending moment is decreased as the load is moved, so that the moment occuring with load 3 at the section will be a maximum and may be the absolute maximum. In this case its value is:

$$(55 + 45 + 35)\tfrac{10}{60} \times 15 + (25 + 15)\tfrac{5}{60} \times 15 - 10 \times 10 = 288 \text{ in.-lb.}$$

The absolute maximum is, therefore, produced when load 2 is at the section and it has a value of 306 in.-lb.

A consideration of the shape of the influence line for moment indicates that the greatest moment on a given span will occur on a section at or near midspan. It may be shown that the greatest moment resulting from a given set of concentrated loads will occur on a section under one of the loads when the center of the span lies halfway between the critical load and the resultant of the set of loads. The demonstration is as follows:

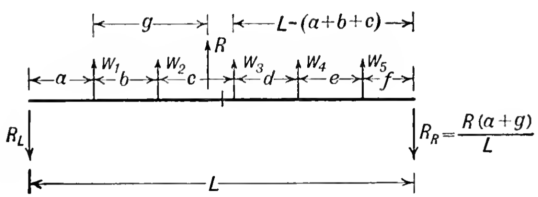


FIG. 3 : 15

Assuming that the magnitudes of the loads shown in Fig. 3 : 15 are such that when load W_3 is located near the center of a span the shear equals or passes through zero under load W_3, the problem resolves itself into determining the location of W_3 that will produce the absolute maximum moment on the span for the given loads. For the loads as shown the moment at W_3 is:

$$M = \frac{-R(a + g)}{L}[L - (a + b + c)] + W_4 d + W_5(d + e)$$

As the loads are moved the value of a increases or decreases with a consequent change in the value of the moment which may be written:

$$\frac{dM}{da} = \frac{-R}{L}(L - 2a - b - c - g)$$

and since, for M to be a maximum its first derivative must be zero, we have,

$$\frac{-R}{L}(L - 2a - b - c - g) = 0$$

whence

$$L - 2a - b - c - g = 0$$

or,

$$a + g = L - (a + b + c)$$

But the distance from the left support to the resultant of the loads is $a + g$ and the distance from the load W_3 to the right support is $L - (a + b + c)$. It is obvious, therefore, that the resultant must be as far from one end of the span as the load under which the shear passes through zero is from the other. This is the same as saying that the distances from the center of the span to the resultant and to the critical load must be equal, or, in other words, that the center of the span must lie halfway between the resultant and the critical load.

The following simple example will be worked out to illustrate the application of this relationship.

Problem — A beam to support a bomb rack is to be attached to the wing spars of a converted commercial airplane so that a 400-lb. bomb may be attached in whatever position may be necessary to maintain the balance of the airplane. The distance between the spars is 60 in. and the loads from a 400-lb. bomb, with a load factor of 7 included, are as shown in Fig. 3 : 16. For what maximum moment must the beam be designed?

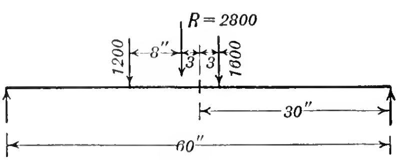


FIG. 3 : 16

To determine which of the loads will probably be critical move the loads until their resultant lies at the center of the span. Then $R_L = R_R = 1400$ lb. The shear just to the left of the 1200-lb. load is $+1400$ and just to the right it is $+200$ so the 1200-lb. load is not critical since the shear does not pass through zero at the section at which it is applied. The center of the span must therefore lie halfway between

the resultant and the 1600-lb. load, making the right reaction equal to

$$\frac{2800(30 - 3)}{60} = 1260 \text{ lb.}$$

and the moment at the section under the 1600 lb. load equal to

$$M = (30 - 3)R_R = \frac{2800(30 - 3)^2}{60} = 34{,}020 \text{ in.-lb.}$$

PROBLEMS

3 : 1. Find the reactions for the wing spar shown in Fig. 3 : 3 assuming a concentrated load of 2000 lb. acting upward at the middle of the 80-in. span, a 500-lb. load acting downward at the outer end of the cantilever.

3 : 2. Draw curves of shear and bending moment for the loading given in the above problem.

3 : 3. Assume another spar to be attached by a pin connection — one which transmits shear but not bending moment — to the extreme right end of the spar shown in Fig. 3 : 3 and to be supported by a vertical strut 50 in. farther to the right. Extend the uniformly distributed load of 20 lb. per in. over this new beam and determine the reactions. Draw curves of shear and bending moment for both spars.

3 : 4. Draw an influence line for the magnitude of H_L, the horizontal component of the left reaction, for the arrangement of spars described in Problem **3 : 3.**

3 : 5. Draw an influence line for bending moment at the middle of the 80-in. span for the spars of Problem **3 : 3.**

3 : 6. A concentrated load, A, precedes another load, B, by a fixed distance of 6 in. 16 in. behind B is a second load A followed at a 6-in. interval by a second load B. If loads A are each 200 lb. and loads B each 300 lb., what is the absolute maximum shear at the quarter point of a simply supported beam 20 in. long? On one 60 in. long? What are the magnitudes of the absolute maximum moments on each of these spans?

CHAPTER IV

CONTINUOUS AND RESTRAINED BEAMS

Wing spars and many of the other members that carry bending in an airplane structure are often supported at more than two points, and when supported at only two points are frequently framed into the supporting members in such a way that a moment is developed and forms a part of at least one of the reactions. Such members are statically indeterminate as regards the outer forces — the reactions — so that recourse must be had to the principle of consistent deformations in order that sufficient equations may be obtained to supplement those of equilibrium and make possible the determination of the reactions. One of the most common applications of this principle is in the form of the assumption that the slope of the tangent to the elastic curve at a section an infinitesimal distance to the left of an intermediate support on a continuous beam is the same as the slope at a section an infinitesimal distance to the right of that support. By the use of this assumption equations commonly called "Three-Moment Equations" may be developed to show the relationship between the moments on sections of the beam over any three consecutive supports and by applying these equations in conjunction with those of statics a complete solution of the problem may be obtained.

4 : 1. Development of the Three-Moment Equation — From mechanics we have the following very important relationships:

$$i = \int \frac{M\,dx}{EI} + C_1$$

$$y = \int i\,dx + C_2$$

where x represents the distance of any section from the left support,
M represents the equation for bending moment at any section,
i represents the slope of the tangent to the elastic curve,
y represents the deflected position of any section, x, with respect to its original position,
E is the modulus of elasticity of the material in the beam,
I is the moment of inertia of any section, x, about an axis through the centroid of the section and normal to the plane of the loads.
C_1 and C_2 are the constants of integration.

In Fig. 4 : 1 two adjacent spans are shown, each as a free body restrained at the ends by the applied moments M_1, M_2, M_2', and M_3, and subjected to uniformly distributed loads of w_1 and w_2 lb. per in. Taking the left reaction on each span as the origin of coördinates and

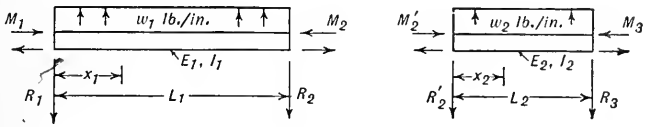


FIG. 4 : 1

assuming E to be constant throughout the length of both spans, I to be constant throughout the length of each span but not the same in both, the following equations may be set up:

Span 1

$$M_{x_1} = M_1 + \frac{(M_2 - M_1)x_1}{L_1} - \frac{w_1 L_1 x_1}{2} + \frac{w_1 x_1^2}{2}$$

$$i_{x_1} = \frac{1}{EI_1}\left[M_1 x_1 + \frac{(M_2 - M_1)x_1^2}{2\,L_1} - \frac{w_1 L_1 x_1^2}{4} + \frac{w_1 x_1^3}{6} + C_1\right]$$

$$y_{x_1} = \frac{1}{EI_1}\left[\frac{M_1 x_1^2}{2} + \frac{(M_2 - M_1)x_1^3}{6\,L_1} - \frac{w_1 L_1 x_1^3}{12} + \frac{w_1 x_1^4}{24} + C_1 x_1 + C_2\right]$$

Span 2

$$M_{x_2} = M_2' + \frac{(M_3 - M_2')x_2}{L_2} - \frac{w_2 L_2 x_2}{2} + \frac{w_2 x_2^2}{2}$$

$$i_{x_2} = \frac{1}{EI_2}\left[M_2' x_2 + \frac{(M_3 - M_2')x_2^2}{2\,L_2} - \frac{w_2 L_2 x_2^2}{4} + \frac{w_2 x_2^3}{6} + C_1'\right]$$

$$y_{x_2} = \frac{1}{EI_2}\left[\frac{M_2' x_2^2}{2} + \frac{(M_3 - M_2')x_2^3}{6\,L_2} - \frac{w_2 L_2 x_2^3}{12} + \frac{w_2 x_2^4}{24} + C_1' x_2 + C_2'\right]$$

If these two separate spans were to be connected at the intermediate support, R_2, to form one continuous beam, M_2 would equal M_2' and the slope of the beam an infinitesimal distance to the left of the intermediate support would be the same as that an infinitesimal distance to the right. Hence $i_{x_1} = i_{x_2}$ when $x_1 = L_1$ in the first span and $x_2 = 0$ in the second. We may therefore equate the expressions for i_{x_1} and i_{x_2} for these conditions but find that if this is done the values of C_1 and C_1', the constants of integration, remain undetermined. For the case under consideration there are no sections in either span at which the slope is known, so these constants cannot be determined from the equations

for slope alone. If it is assumed that the left support deflects a distance y_1 under the given load, the intermediate, a distance y_2, and the right support y_3 the values of C_1, C_1', C_2 and C_2' may be determined from the equations for deflection and substituted in those for slope.

Then
$$C_1 = \frac{EI_1(y_2 - y_1)}{L_1} - \frac{(M_2 + 2\,M_1)L_1}{6} + \frac{w_1L_1^3}{24}$$
$$C_1' = \frac{EI_2(y_3 - y_2)}{L_2} - \frac{(M_3 + 2\,M_2)L_2}{6} + \frac{w_2L_2^3}{24}$$

giving,
$$\frac{1}{EI_1}\left[M_1L_1 + \frac{(M_2 - M_1)L_1}{2} - \frac{w_1L_1^3}{4} + \frac{w_1L_1^3}{6} + \frac{EI_1(y_2 - y_1)}{L_1} - \frac{(M_2 + 2\,M_1)L_1}{6} + \frac{w_1L_1^3}{24}\right] =$$
$$\frac{1}{EI_2}\left[\frac{EI_2(y_3 - y_2)}{L_2} - \frac{(M_3 + 2\,M_2)L_2}{6} + \frac{w_2L_2^3}{24}\right]$$

which reduces to,
$$\frac{M_1L_1}{I_1} + 2\,M_2\left[\frac{L_1}{I_1} + \frac{L_2}{I_2}\right] + \frac{M_3L_2}{I_2} = \frac{w_1L_1^3}{4\,I_1} + \frac{w_2L_2^3}{4\,I_2} + \frac{6\,E(y_1 - y_2)}{L_1} + \frac{6\,E(y_3 - y_2)}{L_2} \qquad 4:1$$

The deflections at the points of support are generally small and may usually be neglected. In some cases, however, their effect is appreciable and it should always be provided for when precise results are desired.

It is to be noted that the terms on the left side of the equation involve the ratios of length to stiffness for the two spans as well as the bending moment over each support but are independent of the loads. The terms which involve the loads are all on the right side of the equation in this case where the load has been taken to be uniformly distributed. It may be shown that this is true for all types of loading, the left side of the equation being the same in each case, so that provision may be made for one or more loads, or one or more types of load distribution, by setting up the left side of the equation as it is given in Formula 4 : 1 and writing sufficient terms on the right side to provide for each load in each span. Figure 4 : 2 gives the terms that are to be used on the right side of the equation for various types of loading. When the load is applied to the left of the two spans under consideration the loading terms involving w_1L_1, etc., are used; when in the right span, those involving w_2L_2.

LOADING TERMS FOR RIGHT SIDE OF THREE-MOMENT EQUATION

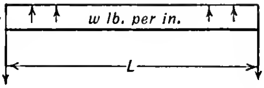
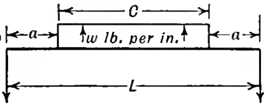
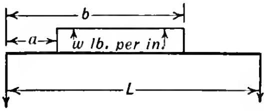
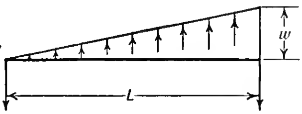
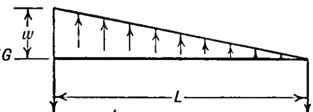
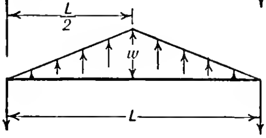
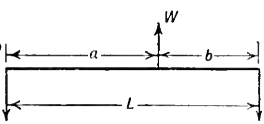
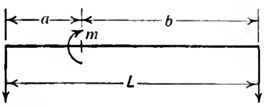
TYPE OF LOADING		IN LEFT BAY	IN RIGHT BAY
UNIFORMLY		$+\frac{w_1L_1^3}{4I_1}$	$+\frac{w_2L_2^3}{4I_2}$
DISTRIBUTED		$+\frac{w_1c_1(3L_1^2-c_1^2)}{8I_1}$	$+\frac{w_2c_2(3L_2^2-c_2^2)}{8I_2}$
LOADS		$+\frac{w_1[b_1^2(2L_1^2-b_1^2)-a_1^2(2L_1^2-a_1^2)]}{4I_1L_1}$	$+\frac{w_2[b_2^2(2L_2-b_2)^2-a_2^2(2L_2-a_2)^2]}{4I_2L_2}$
UNIFORMLY		$+\frac{2w_1L_1^3}{15I_1}$	$+\frac{7w_2L_2^3}{60I_2}$
VARYING		$+\frac{7w_1L_1^3}{60I_1}$	$+\frac{2w_2L_2^3}{15I_2}$
LOADS		$+\frac{5w_1L_1^3}{32I_1}$	$+\frac{5w_2L_2^3}{32I_2}$
CONCENTRATED LOAD		$+\frac{W_1a_1(L_1^2-a_1^2)}{I_1L_1}$	$+\frac{W_2b_2(L_2^2-b_2^2)}{I_2L_2}$
EXTERNAL MOMENT		$+\frac{m_1}{I_1}\left(\frac{3a_1^2}{L_1}-L_1\right)$	$+\frac{m_2}{I_2}\left(L_2-\frac{3b_2^2}{L_2}\right)$
GENERAL FORM		$+\frac{L_1^2}{I_1}\int\left[\frac{a_1}{L_1}-\left(\frac{a_1}{L_1}\right)^3\right]w_1\,da_1$	$+\frac{L_2^2}{I_2}\int\left[\frac{L_2-a_2}{L_2}-\left(\frac{L_2-a_2}{L_2}\right)^3\right]w_2\,da_2$

Fig. 4 : 2

4 : 2. Application of the Three-Moment Equation — In Fig. 4 : 3 we have a beam supported at four points and subjected to a series of vertical loads. There are, then, four reactions, R_1, R_2, R_3, R_4, and, since there is no horizontal load, there are only two equations of statics: $\Sigma V = 0$ and $\Sigma M = 0$. It is therefore necessary to obtain two more equations in order to determine the reactions. These can be obtained by applying the three-moment equation twice, first to the spans between reactions 1 and 3 and second to those between reactions 2 and 4. It is to be noted that M_1, the bending moment at R_1, may be determined from the load on the cantilever section to the left of R_1, giving $M_1 = +500$ in.-lb. M_4 is zero since the beam ends at R_4 and there are no horizontal forces or external moments applied at that section. Assum-

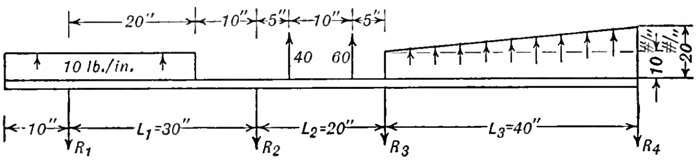


FIG. 4 : 3

ing that the beam is of uniform cross-section throughout its length: $I_1 = I_2$, and that the supports do not deflect: $y_1 = y_2 = y_3 = y_4 = 0$, we have, by using Fig. 4 : 2:

For spans 1 and 2,

$$M_1L_1 + 2\,M_2(L_1 + L_2) + M_3L_2 = \frac{w_1[b_1^2(2\,L_1^2 - b_1^2) - a_1^2(2\,L_1^2 - a_1^2)]}{4\,L_1} + \frac{W_2b_2(L_2^2 - b_2^2)}{L_2} + \frac{W_2'b_2'[L_2^2 - (b_2')^2]}{L_2}$$

which gives,

$$+500 \times 30 + 2\,M_2(30 + 20) + M_3 \times 20 = \frac{10[400(1800 - 400)]}{4 \times 30} + \frac{40 \times 15(400 - 225)}{20} + \frac{60 \times 5(400 - 25)}{20}$$

Whence,

$$100\,M_2 + 20\,M_3 = 42{,}550.$$

For spans 2 and 3,

$$M_2L_2 + 2\,M_3(L_2 + L_3) + M_4L_3 = \frac{W_2a_2(L_2^2 - a_2^2)}{L_2} + \frac{W_2'a_2'[L_2^2 - (a_2')^2]}{L_2} + \frac{w_3L_3^3}{4} + \frac{7\,w_3'L_3^3}{60}$$

which gives,

$$M_2 \times 20 + 2\,M_3(20 + 40) + 0 = \frac{40 \times 5(400 - 25)}{20} + \frac{60 \times 15(400 - 225)}{20} + \frac{10 \times 40^3}{4} + \frac{7 \times 40^3 \times 10}{60}$$

Whence,

$$20\,M_2 + 120\,M_3 = 246{,}300.$$

Solving these two equations for M_2 and M_3 gives $M_2 = 16$ in.-lb. and $M_3 = 2045$ in.-lb.

Although the three-moment equations have now been solved, the reactions have not yet been determined and, apparently, little has been gained. It is to be noted, however, that sufficient data are now available to permit isolating each span and determining the reactions on it. The right reaction for the first span taken by itself is the same as the shear just to the left of R_2 for the beam as a whole. The left reaction for the second span alone is the shear just to the right of R_2 for the beam as a whole. The sum of these shears, having due regard for their signs, will therefore give us the reaction, R_2. The shear just to the left of a reaction will be represented by S with a negative subscript, that to the right, by S with a positive subscript. For the above beam,

Cantilever,	$S_{-1} =$	$+100.0$	
From span 1,	$S_{+1} = -\dfrac{10 \times 20 \times 20}{30} + \dfrac{16 - 500}{30}$	$= -149.4$	$R_1 = 249.4$
From span 1,	$S_{-2} = +\dfrac{10 \times 20 \times 10}{30} + \dfrac{16 - 500}{30}$	$= +\ 50.6$	
			$R_2 = -5.4$
From span 2,	$S_{+2} = -\dfrac{40 \times 15}{20} - \dfrac{60 \times 5}{20} + \dfrac{2045 - 16}{20}$	$= +\ 56.0$	
From span 2,	$S_{-3} = +\dfrac{40 \times 5}{20} + \dfrac{60 \times 15}{20} + \dfrac{2045 - 16}{20}$	$= +156.0$	
			$R_3 = 472.8$
From span 3,	$S_{+3} = -\dfrac{10 \times 40 \times 20}{40} - \dfrac{10 \times 40}{6} + \dfrac{-2045}{40}$	$= -316.8$	
From span 3,	$S_{-4} = +\dfrac{10 \times 40 \times 20}{40} + \dfrac{10 \times 40}{3} + \dfrac{-2045}{40}$	$= +283.2$	$R_4 = 283.2$
Right end,	$S_{+4} =$	0	

The reactions, the moments and the shears at each point of support have now been determined. By using the relationships developed in Chapter III it is possible to determine the shear or bending moment

on any section between any two of the supports. The entire length of the beam may therefore be investigated for purposes of design.

4 : 3. Beams with Fixed Ends — It is sometimes desirable to reduce the deflection of a beam by attaching it to the members which support it in such a way that bending will be developed at the points of attachment when the beam deflects under load. A good example of this is illustrated by the rear fin spar of Fig. 4 : 4 which is inserted in the tail post at its lower end and braced by wires or struts near its upper end. The combination of fin spar and tail post might be considered as a single continuous beam, but even so a precise determination of the bending-moment curve for the unit would be practically impossible as the tail post would normally be connected to both upper and lower longerons in such a manner that unknown moments would form part of its reactions.

If the rear fin spar is relatively light as compared to the tail post, and it often is, it is permissible to assume that its lower end is "fixed," i.e., that the slope of the elastic curve is not changed by the application of the load. This assumption would be strictly true only in case the tail post were of infinite stiffness, but its use in practical cases makes it possible to determine limiting values for the bending moment at the base of the rear fin spar. If we apply the assumption that the slope of the elastic curve at the left end of a span is zero to the slope equation of Art. 4 : 1, we get an expression identical with that which would have been obtained if we had assumed that the beam was continuous with a span of zero length to the left of the fixed point of support. Thus, to analyze the case of the rear fin spar of Fig. 4 : 4 we would proceed as follows:

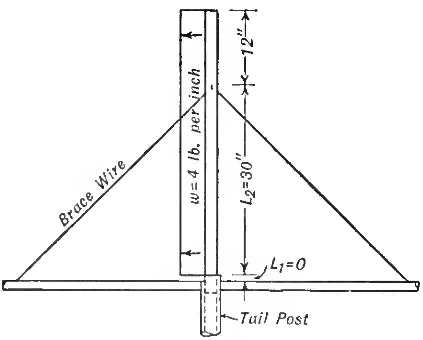


Fig. 4 : 4

Assume the beam to be supported at three points. The lowest point of support, R_1, is an infinitesimal distance below the top of the tail post. At this section M_1 will be assumed equal to zero. The second point of support, R_2, is at the top of the tail post where the fin spar is inserted. The third support, R_3, is furnished by the brace wires which are attached to the stabilizer spars. At R_3 the moment, M_3, due to the cantilever span is $\frac{4 \times 12^2}{2} = 288$ in.-lb. Then, since

$I_1 = I_2$ and M_1, L_1 and $w_1 = 0$,

$$2\,M_2L_2 + M_3L_2 = \frac{w_2L_2^3}{4}$$

$$2\,M_2 \times 30 = \frac{4 \times 30^3}{4} - 288 \times 30$$

$M_2 = 306$ in.-lb., bending moment at the top of the tail post.

With this moment known the shears and reactions on the fin spar may be determined and curves of shear and moment drawn for use in designing the spar itself.

In practice the condition of zero change in slope at a support is never completely attained. When it is assumed, care should be taken that some allowance should be made for the error involved. In the case used for the illustration, the moment at the upper supports of the fin post is not affected by the assumption. The moment at the lower end will probably be a little smaller than that computed as the tail post would give in such a manner as to decrease this moment. If both members are of approximately the same stiffness, the error may be quite large but will be on the safe side. A decrease in the bending at the lower end, however, will involve an increase in the bending moment of opposite sign between the two supports, so that for design purposes a reasonable allowance should be made for this effect. The safest allowance would be to compute two moment curves, one in which the lower end of the fin spar was assumed fixed and the other in which it was assumed pinned, and to compute the margin of safety of any given point on the basis of the larger of the two bending moments computed for that point.

4 : 4. Effect of an External Moment Applied at an Intermediate Support — Airplane designers occasionally use eccentric fittings the effect of which is to produce bending moments at the sections where the fittings are attached. If such fittings were to be used at the intermediate support of a beam continuous over two spans it is obvious that the moment just to the left of the support would not be equal to that just to the right, or $M_{-2} \neq M_{+2}$. The left side of the three-moment equation may be arranged to provide for a moment M_e at the intermediate support by writing,

$$\frac{M_1L_1}{I_1} + \frac{2\,M_{-2}L_1}{I_1} + \frac{2\,M_{+2}L_2}{I_2} + \frac{M_3L_2}{I_2} \quad \text{where} \quad M_{+2} = M_{-2} + M_e.$$

4 : 5. Effect of Variation in Moment of Inertia — The three-moment equation is developed under the assumption that the moment of inertia of the member under consideration is constant throughout each span

although it may vary between spans. This equation cannot be applied to beams that taper or have changes in moment of inertia within any span unless an average value of I is used. In such cases the results will be approximate, the degree of approximation depending on the magnitude of the variation in the moment of inertia. Where precise results are desired one of the methods based on the deflection of the beam as described in a later chapter must be used.

4 : 6. Effect of Axial Loads — The foregoing equations have been developed under the assumption that the loads were all normal to a plane passing through the neutral axis of each section of the beam so that no axial tensions or compressions were developed. The effect of an axial tensile load on a beam which is deflected under a lateral load is to decrease the bending moment due to the lateral load alone. An axial compression, on the other hand, tends to increase the bending moment at each section, an effect which is discussed and provided for in Chapter XI.

PROBLEMS

4 : 1. Prove the validity of formula 4 : 1 in this chapter showing all steps of the computation omitted in the text.

4 : 2. Prove the validity of the method outlined above for the treatment of beams with fixed ends, by making the substitutions suggested in the text.

4 : 3. Draw curves of shear and bending moment for the beam shown in Fig. 4 : 3 assuming it to be turned end for end so that the present R_4 becomes the new R_1, L_3 becomes L_1, etc.

CHAPTER V

DESIGN OF SIMPLE BEAMS

The design of any structure is made in three steps: the determination of the external loads, the computation of the internal stresses produced by those loads on the members which are to carry them, and the determination of the allowable internal stresses in those members. These steps are of equal importance, and the safety of the structure depends on all three being properly executed. In all but the simplest design problems it is necessary to make trial designs, computing the stresses that the external loads would impose on each, comparing these with the allowable stresses for the material used, and rejecting the design if the imposed stresses are too high. Even though the stresses imposed on the members of the trial design are lower than the allowable, that is not necessarily the solution of the design problem, as a lighter or a cheaper structure might be devised that would also satisfy this criterion. The business of the designer being to produce the best design for a given purpose, not simply to adopt any design that will do the work, the process of designing normally consists in the making of a number of trials until the engineer feels confident that he has developed one that cannot be improved upon.

In a few cases where the loadings are simple and the type of structure to be used has been standardized, a member can be designed directly, i.e., without investigating a number of trial designs. This is true where the standardized members are rated according to the loads they will safely carry. In other cases, once the type of member to be used has been decided on, it is possible to compute the theoretical minimum allowable size and choose the next larger practical one.

One type of member that must often be designed by the aeronautical engineer is the beam subjected to pure flexure, i.e., bending not accompanied by loads or components parallel to the axis of the beam. The required size of each cross-section of such a beam depends on the bending moments and shears occurring at that section. The methods used for computing these bending moments and shears have been described in Chapters III and IV. In this chapter it will be shown how, these factors being known, the size of member necessary can be determined. For clarity this will be done by the use of two numerical examples, one in which the design will be made directly, and another in which the normal method of design must be followed.

5 : 1. Design of Straight Axle — The design problem to be handled by the direct method is that of the axle for the Vee type of chassis shown in front elevation in Fig. 5 : 1. This axle is a single straight round tube carrying two 28 x 4-in. wheels at its ends and supported by shock absorber cord at points A and B. The design ground reaction on each wheel is 7150 lb., this being about the proper value for a 2200-lb. airplane designed for a load factor in landing of 6.5.

Curves of load, shear, and bending moment on the axle are shown in Fig. 5 : 2. The only external loads are the 7150-lb. loads acting up at the center-lines of the wheels, and reactions acting down at A and B where the axle is connected to the remainder of the chassis by the shock absorber cord. The weight of the axle itself will be so small in comparison with these loads that it may be neglected. The computations of the values for the reactions and the shear and bending moment curves should need no further comment.

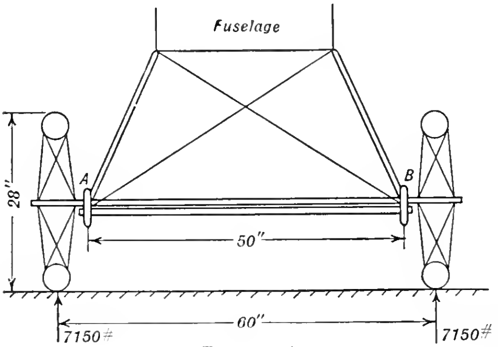


FIG. 5 : 1

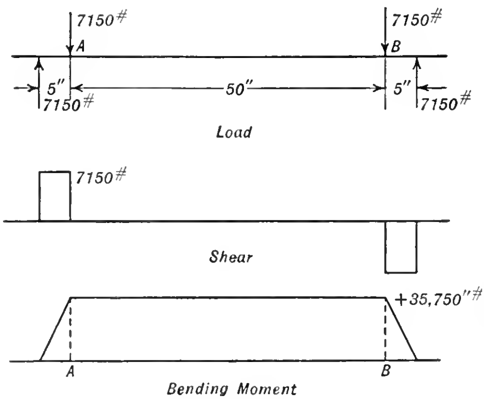


FIG. 5 : 2

On account of the dimensions of the hub of the standard 28 × 4 wheel, it will be assumed that the axle will be a 1 3/4-in. O.D. round tube if one can be found that will carry the load. As axles are normally made of alloy steel tubing heat-treated to an ultimate tensile strength of 180,000 lb. per sq. in., it will be assumed that such material will be used in this case.

Texts on Mechanics of Materials[1] show that in a case of simple

[1] Fuller and Johnson, "Applied Mechanics," Vol. II, page 108. Swain, "Strength of Materials," page 183. Seeley, "Resistance of Materials," Art. 34. Poorman, "Strength of Materials," pages 92 to 97. Morley, "Strength of Materials," Arts. 60–64 inc.

bending the unit normal stress of tension or compression at any point on a cross-section normal to the axis of the beam may be obtained from the formula

$$f = \frac{My}{I} \qquad 5:1$$

where f = unit normal stress in lb. per sq. in.
M = bending moment on the section in inch pounds.
I = moment of inertia of the section in inches.[4]
y = distance, parallel to the plane of bending, from the centroid of the section.

Study of this formula brings out that at all points on a line through the centroid and perpendicular to the plane of bending, $y = 0$, and therefore the normal stress along this line is zero. This line is therefore called the "neutral axis." If the conventions for signs stated in Art. 3 : 6 are followed, and the bending moment is positive, the unit stress at all points above the neutral axis will be compressive, and vice versa.

On any given section normal to the axis of the beam, the maximum unit stress will be found at the "outermost fiber," i.e., the point on the section for which y, the distance from the neutral axis, is a maximum. If all normal sections are identical, as would be true in this case where the member being investigated is a straight round tube, the maximum unit stress will evidently be found at the outermost fiber of the section at which the bending moment is a maximum. A glance at the bending moment curve of Fig. 5 : 2 shows that the maximum bending moment is 35,750 in.-lb., and that it acts on all sections between A and B, any of which may therefore be regarded as the critical one.

In round steel tubes subjected to pure flexure the modulus of rupture or allowable value of the normal stress, f, of Formula 5 : 1, may be taken as the equal to the ultimate tensile strength, in this case 180,000 lb. per sq. in. Substituting this value and the maximum moment in Formula 5 : 1, we find the minimum allowable value of I/y for this case to be 35,750/180,000 = 0.1985. From Table 9 : 5, page 126, in which the I/y values for round tubes are listed, it is found that the thinnest walled 1 3/4-in. tube that may be used is the 1 3/4 × 0.120 for which $I/y = 0.2345$. This size would probably be chosen unless other factors entering the design problem made it unsatisfactory.

The reader may have noted that the only stresses that were investigated above were those on sections normal to the axis of the member, and he may suspect that even larger stresses might be found on sections at other angles to the axis. For certain types of loading, such as combined bending and torsion, the maximum stresses would be found

on sections not at right angles to the axis, but, as has been demonstrated in the study of Mechanics of Materials, this cannot happen in a beam subjected to pure flexure. In this type of beam the maximum normal stress is always that at the outermost fiber of a section at right angles to the beam axis.

In working out the above problem, it was assumed that the axle was of constant cross-section and that the load from each wheel was applied to the axle as a concentrated load directly above the center of the area of contact of the tire with the ground. In actual practice, the axle tube would be ground down for some 6 in. from each end to an outside diameter of 1.688 in. to provide a smooth bearing for the hub of the wheel, and the wheel load would be distributed along the length of the hub. This would cause the outer portion of the actual bending moment curve to be about as shown by the solid line in Fig. 5 : 3. Instead of a straight line from zero at the point directly above the center of the area of contact of the wheel with the ground to a maximum at point A where the axle is supported by the shock absorber cord, the curve would start from zero at C, the outer end of the wheel hub, and meet the curve first described at D, the inner end of the hub, and then be identical with it from D to A. The shape of the curve between C and D would depend on the distribution of the transfer of load from the hub to the axle, but would be of no special interest to the designer unless he suspected that there would be a weak section between these points. Similarly the fact that the "point" of support at A cannot be a true point, since the supporting effect of the shock absorber cord must extend over a finite length of the axle, would prevent the true bending moment curve from having the sharp changes in direction shown at A and B in Fig. 5 : 2. As this would not affect the bending moment on most of the axle between A and B, however, it would not affect the design procedure.

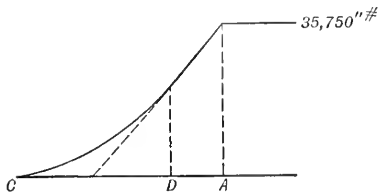


FIG. 5 : 3

If the ground down portion of the axle were to extend to point A, it would be necessary to use the section modulus of that section for computing the maximum imposed stress. Usually, however, it will stop just inside of D, the inner end of the wheel hub. At this point, the bending moment will be less than between A and B, but so will the section modulus, hence the imposed unit stress may be greater. If the wall thickness before grinding down to 1.688 in. outside diameter was 0.120 in., the thickness of the ground portion would be only 0.089 in.,

which might be too small for the loads to which the inner end of the ground portion would be subjected. If this were the case it would be necessary to use a tube with a thicker wall. Essentially the same methods of computation would be used to investigate this phase of the problem as were employed above.

If the section is symmetrical about the neutral axis, the maximum values of y on each side of that line are equal and the maximum tension and compression unit stresses will be numerically equal. If it is unsymmetrical the maximum values of y will normally be unequal and the maximum stresses of the two types will be unequal. With a material like steel where the modulus of rupture is assumed to be the same for both the tension and the compression sides of the beam, the larger of the two computed stresses would be used in determining the margin of safety. In wood, the modulus of rupture for the tension side is larger than that for the compression side, so if the maximum value of y on the compression side of the neutral axis is smaller than that on the tension side, both points must be investigated. An example of this is given in the next article.

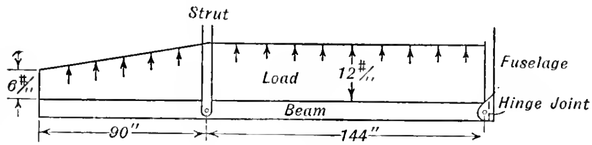


FIG. 5 : 4

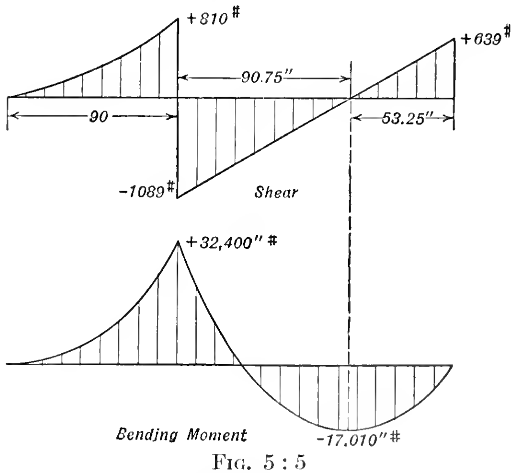


FIG. 5 : 5

5 : 2. Design of Wood Beam — A wood beam not to exceed 6.5 in. in depth is to be designed to carry the loads shown in Fig. 5 : 4. This is a slight simplification of the problem of the design of a rear lower wing spar for a biplane with wings of 90-in. chord and Clark Y airfoil section, for Low Angle of Attack.

The curves of shear and bending moment for this beam, which is assumed hinged at its connection to the fuselage, are shown in Fig. 5 : 5.

The first trial design will be a spruce box beam with unequal chords

and two ply spruce plywood webs, the grain of the plies being at an angle of 45 degrees to the axis of the member. The dimensions of the section chosen for the first trial are shown in Fig. 5 : 6.

As the bending moments and shears are known, the first step is to determine the area, moment of inertia, and section moduli of the cross-section. In computing these quantities, the effect of the plywood web will be neglected.[1] As these computations can be considerably simplified by arranging them in a suitable form, a suggested method is used here. The first step is to draw a diagram of the cross-section approximately to scale, and put on it the known dimensions. In this case, these are the total width, 1.50 in.; the width of flanges, 1.25 in.; the total depth, 6.50 in.; the thicknesses of the flanges, 1.00 in. for the upper and 0.50 in. for the lower; and the clear distance between flanges, 5.00 in. The next step is to compute the sectional area and the distance from the lower surface to the neutral axis and centroid of the section. The computations of these steps may be written down as follows:

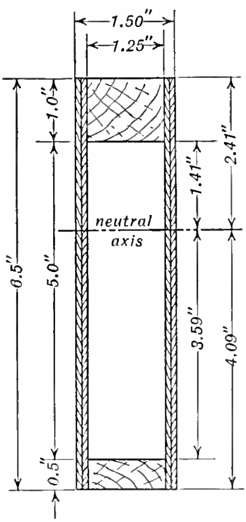


FIG. 5 : 6

$$
\begin{array}{lcrcrcl}
1.25 \times 1.00 & = & 1.250 & \times & 6.00 & = & 7.500 \\
1.25 \times 0.50 & = & \underline{0.625} & \times & 0.25 & = & \underline{0.156} \\
\text{Area} & = & 1.875 & & m & = & 7.656
\end{array}
$$

$$
\text{n.a. distance} = \frac{7.656}{1.875} = 4.09 \text{ in. above lower surface.}
$$

In the above computations the widths of the flanges are written down in the first column, and the thicknesses in the second. The products in the third column are obviously the areas of the flanges. The quantity in the fourth column is the distance from the line of reference, here the lower surface of the beam, to the centroid of the area to be multiplied by it, and the product in the fifth column gives the moment of that area about the line of reference. The sum of the values in the third column is the area of the section, and that of the

[1] The Department of Commerce allows one-half of the web material to be considered in the computation of the properties of the sections in this type of construction. In all but very deep beams the increase in properties so obtained is so small that it is hardly worth the added labor of computation. It is considered preferable to use the conservative method of neglecting the web when computing the properties of box spar sections.

fifth column the moment of the total area about the point of reference. The quotient of this moment divided by the area gives the distance from the line of reference to the neutral axis and centroid of the section.

The distance from the neutral axis to the lower surface of the beam having been computed, the distances to the upper surface of the beam and the inner surfaces of the flanges can be found and entered on the drawing of the beam section. It now remains to determine the moment of inertia about the neutral axis, and the section moduli. The simplest method of computing the moment of inertia is to assume the section made up of positive and negative rectangles and use the formula $I = bd^3/3$ for the moment of inertia of a rectangle about its base. These computations would then be written

$$\begin{aligned} I &= \frac{1.25}{3}(2.41^3 + 4.09^3 - 1.41^3 - 3.59^3) \\ &= \frac{1.25}{3}(13.998 + 68.418 - 2.803 - 46.268) \\ &= \frac{1.25}{3} \times 33.345 = 13.90. \end{aligned}$$

In making this computation, the cubes can be found most easily from a table such as is included in almost any engineers' handbook. The section moduli are

$$I/y_u = 13.90/2.41 = 5.77 \text{ in.}^3$$

and

$$I/y_l = 13.90/4.09 = 3.40 \text{ in.}^3$$

The maximum bending moment on the beam is at the outer support and is 32,400 in.-lb. The maximum unit stresses on this section are

$$f_c = My_u/I = 32{,}400/5.77 = 5{,}620 \text{ lb. per sq. in. compression and}$$
$$f_t = My_l/I = 32{,}400/3.40 = 9{,}530 \text{ lb. per sq. in. tension.}$$

The modulus of rupture for this section with the larger flange member in compression may be taken as 6150 lb. per sq. in. for the compression side and 10,000 lb. per sq. in. for the tension side.[1] The margins of safety are therefore $\frac{6150}{5620} - 1 = +0.094$ or $+9.4$ per cent for the compression side, and $\frac{10000}{9530} - 1 = +0.049$ or $+4.9$ per cent for the tension side, both of which are satisfactory.

If the beam were of constant and symmetrical section, it would not be necessary to investigate the stresses due to bending moment at any

[1] The method of computing the modulus of rupture for wood box and I sections is described in Art. 12 : 1, page 239.

other section except that at which the bending moment is a maximum. In this case the section is constant but not symmetrical, and the section of maximum negative bending moment must be investigated as well as that of maximum positive moment. At the section just investigated, it may be noticed that the maximum compressive stress is less than the maximum tension, on account of the neutral axis being nearer to the top of the beam than to the bottom. Also the maximum allowable compression was less than the maximum allowable tension. Near the center of the span where the bending moment is − 17,010 in.-lb. the lower flange will be in compression and the maximum stress on it will be greater than the maximum tension on the upper. Even though the total bending moment on this section is less than at the strut point, the maximum compression on this flange may be greater than the allowable, and this must be investigated.

Using the same procedure as above the maximum compression at the point of maximum moment in the span will be

$$f_c = My_l/I = 17{,}010/3.40 = 5{,}000 \text{ lb. per sq. in.}$$

With the smaller flange in compression, the allowable compressive stress is only 5700 lb. per sq. in. This value is less than that used before because the ratio of the thickness of the compressive flange, in this case the lower, is less than that of the thickness of the upper flange to the total height of the beam. The margin of safety is therefore $\frac{5700}{5000} - 1 = +0.140$, or $+14$ per cent. It should be noted that while this margin of safety is larger than the margin of safety in compression at the strut point, the difference is much less than is indicated by the ratio of the bending moments at the two sections.

It is not necessary to investigate the margin of safety on the tension side of the beam at the section between the supports in this case, as the maximum stress in tension is sure to be less than at the strut since the bending moment is smaller, and the section modulus to be used larger, while the allowable stress in tension remains the same. The margin of safety in tension is therefore certain to be much larger in the span than at the strut point.

If the trial design had been not of constant but of varying section, it would have been necessary to investigate all sections where it might be found that the unit stress, i.e., the ratio of the bending moment on the section to the section modulus, might be greater than the allowable stress of the character being computed. In such cases the inexperienced designer will do well to compute the margins of safety at a number of

points along the beam, being careful to include any points at which there is a sharp change either in the curve of bending moments or in the dimensions of the sections. As he gains experience, he will learn to pick the critical sections without many trials, and thus expedite his work.

In the case of a tapered wing beam of a cantilever monoplane the bending moment increases from zero at the wing tip to a maximum at the point of connection to the fuselage. The moment of inertia of the beam also increases from the tip to the fuselage. The designer should not be content with investigating the margin of safety at the connection of the wings to the fuselage, but should also investigate several sections between that point and the wing tip, and sometimes he will find that though the margin at the fuselage is positive, the taper of the beam is too pronounced and the margin at some of the intermediate points is negative, and the design therefore unsatisfactory.

5 : 3. Investigation of Wing Beam under Shear Stresses — In the last two articles the beams were investigated for the stresses due to bending only, those due to shear being neglected. In the case of the tube subjected to simple bending, this procedure was proper since for such a member, unless it is very short, the shear load is not likely to be critical. The case of the wood box beam is different and such members should always be investigated in shear.

Texts on Mechanics of Materials give the following formula for the distribution of shear stress over a cross-section of a beam subjected to bending:[1]

$$f_s = \frac{SQ}{bI} \qquad 5:2$$

where f_s = unit shear stress in pounds per square inch.
S = total shear on the section in pounds.
b = width of the section at the point investigated, in inches.
I = moment of inertia of the entire section in inches.[4]
Q = static moment about the neutral axis of the part of the section above or below the point investigated. Q will be numerically the same regardless of whether the part above or the part below the point being investigated is used in the computations, since, by definition, the total moment about the neutral axis must be zero.

[1] Fuller and Johnson, "Applied Mechanics," Vol. II, pages 161 to 169. Swain, "Strength of Materials," page 189. Seeley, "Resistance of Materials," Art. 40. Poorman, "Strength of Materials," Art. 55. Morley, "Strength of Materials," Art. 71.

In any given section, the static moment Q will be zero at the outermost points and will increase to a maximum at the neutral axis, and if the width, b, is a constant, as in a rectangular section, the unit shear will vary in proportion.

In a box section like that shown in Fig. 5 : 6, it should be evident that the maximum shear stress will be found at the neutral axis, as Q is a maximum and b a minimum at this point. For this beam, if the web is neglected, Q at the neutral axis will be $1.25 \times 1.00 \times 1.91 = 2.39$ in.[3] The width of the section may be taken as the total thickness of plywood webs for a box beam — since the webs carry the shear they must be considered here — or that of the solid web of an I-beam. In this case, it is therefore 0.25 in.

Applying Formula 5 : 2, the unit shear stress is

$$f_s = \frac{SQ}{bI} = \frac{1089 \times 2.39}{0.25 \times 13.90} = 749 \text{ lb. per sq. in.}$$

The allowable shear depends on several factors. In the case of a solid spruce rectangle or I-beam, it may be taken as 750 lb. per sq. in. In the case of a box with plywood sides, it depends not only on the species of wood used in the plywood but also on the distance between the centroids of the beam flanges, and the spacing of vertical web stiffeners. These web stiffeners are normally diaphragms of spruce about 1/4 or 3/8 in. in thickness, often with lightening holes. This type of construction was studied at McCook Field by Roy A. Miller. He developed the empirical formula first published in Air Corps I. C. 587 and later adopted by the Department of Commerce, viz.:

$$F_s = 960 + \frac{3140}{\sqrt[3]{C}} - 45.5\,D \qquad 5:3$$

where F_s = the allowable shear stress in pounds per square inch for spruce or mahogany plywoods.

C = the center to center spacing of stiffeners or diaphragms in inches.

D = the distance between the centroids of the top and bottom flanges of the beam in inches.

As Mr. Miller's tests showed that the allowable unit stress for webs of beams without stiffeners was given by using the formula for a value of 32 in. for C, and the unit stress developed in the trial design is small, that design will be investigated for this stiffener spacing to see if any

stiffeners are needed. Substituting the proper values in Formula 5 : 3,

$$F_s = 960 + \frac{3140}{\sqrt[3]{32}} - 45.5 \times 5.75 = 960 + 990 - 262$$
$$= 1688 \text{ lb. per sq. in.}$$

This beam would therefore be satisfactory in shear even though no stiffeners were used, since it has a margin of safety of $\frac{1688}{749} - 1 = +1.25$ or +125 per cent.

It may be noted that the shear used for this investigation is that just inside the outer support, though according to Fig. 5 : 5 the shear changes at that section from +810 to −1089 lb., a total change of 1899 lb. It might be thought that on this account the beam should be designed to carry a shear of 1899 lb. instead of only 1089 lb. This is not necessary, however, as the support is not at a single point, but extends over a finite area, along which the shear changes more gradually, so that it is not 1899 lb. on any one section. Also a filler block would be placed between the flanges to help distribute the load from the strut over the spar section, and this would greatly strengthen the member locally. The maximum shear that the section shown in Fig. 5 : 6 would actually be called on to carry would therefore be something less than even the 1089 lb. on which the margin of safety computations were based, but the small difference is not worth trying to take into account.

These computations complete those necessary to determine whether a beam of the major dimensions chosen for the trial and proper detailed design would be satisfactory. As all the margins of safety found were positive it passes this test, but some of the positive margins were so large that the designer would be very likely to try out a slightly smaller beam, the obvious place from which material might be taken being the web, which turned out to be more than twice as strong as was necessary. It generally happens that some of the margins of safety are negative, in which case more trial designs must be made to find a section that will be satisfactory. Since the method of investigating these additional trial designs will be the same as that used in the computations above, it is not considered necessary to go further into this phase of the problem.

Once the major dimensions of a beam section have been determined, the design of the beam has not, however, been completed. Investigations must still be made of the strength of the glued joints between members of wood beams, or of riveted joints between parts of built up metal beams. Also, when either the external loads or reactions are

concentrated the beam must be designed to have extra strength locally so these loads can be distributed over the cross-section of the beam. These phases of the design, however, require a more extensive knowledge than the student has at this point, so will not be treated here, but will be taken up in later chapters.

PROBLEMS

5 : 1. Determine the wall thickness needed for the axle of Art. 5 : 1 if the outside diameter is ground down to 1.688 in. from the outer ends to points 1.0 in. outboard of the points A and B where the axle is connected to the shock absorber cord.

5 : 2. Assume the conditions stipulated in Problem **5 : 1** and design an axle tube of nickel steel heat-treated to 125,000 lb. per sq. in. ultimate tensile strength.

5 : 3. Design a spruce spar of rectangular cross-section to withstand the same bending moments and shears as those imposed on the box section shown in Fig. 5 : 6. The overall depth of the rectangular section is to be 6.5 in. The allowable stress on the extreme fiber carrying compression will be 9400 lb. per sq. in., the allowable shear parallel to the grain, 750 lb. per sq. in. Compare the two sections as to cross-sectional area, and weight per foot of length.

CHAPTER VI

TORSION

When the line of action of a load does not intersect the axis of the member to which the load is applied, twisting or torsion is developed in the member. Torsional stresses are produced in many of the members used in an airplane, particularly in the main members of movable control surfaces. In such surfaces the member is placed at or near the leading edge while the load is distributed over the entire surface and carried by the ribs which, being cantilevered from the main member, tend to twist it about its longitudinal axis.

Torque tubes are frequently used in other parts of the control system to transmit loads from the control stick to the bell cranks to which the cables or push rods which actuate the surfaces themselves are attached. The use of brakes on landing gear wheels imposes torsional loads on the axle tubes and other members of the chassis structure.

In general, members carrying torsion are made of drawn seamless round tubing of steel or aluminum alloy although sections of other shapes and materials are occasionally used. When used, their design must be checked by tests, as the data on allowable stresses for other than circular shapes and other materials are so meager as to be of little value in general design. For that reason this chapter is confined almost entirely to a consideration of round and tubular sections.

6 : 1. Torsional Stresses in a Circular Bar — Consider a straight bar of uniform circular cross-section and homogeneous material that is loaded by two couples applied in planes perpendicular to the axis of the bar and having equal magnitudes but acting in opposite directions as shown in Fig. 6 : 1. Each cross-section of the bar will then be subjected to shearing stresses which must be determined.

A line AB which is drawn on the surface of the cylinder before the equal and opposite couples, T and $-T$, are applied will move to position AB' due to the rotation of each cross-section of the shaft under the twisting action of the couples. Assuming the end of the bar at A to be fixed so that it cannot rotate, the rotation of each section along the bar will be proportional to its distance, L, from A. It is assumed that any radial line, as OB, remains straight during the distortion of the bar and hence undergoes angular displacement only. Then OB is rotated through an angle, i, to position OB', and BB' represents the

total strain on the circumference of the bar in a length, L, due to the shearing action of the couple, T. Thus the strain on a section at a unit distance from A would be, $e_s = \frac{BB'}{AB} = \frac{Ri}{L}$. If we represent the shearing modulus of elasticity, or modulus of rigidity as it is often called, by G and assume that the law of proportionality of shearing stress to shearing strain applies, then $f_s' = e_sG = \frac{GRi}{L}$ acting tangent to the surface of the cylinder, hence normal to the radius, R. At any

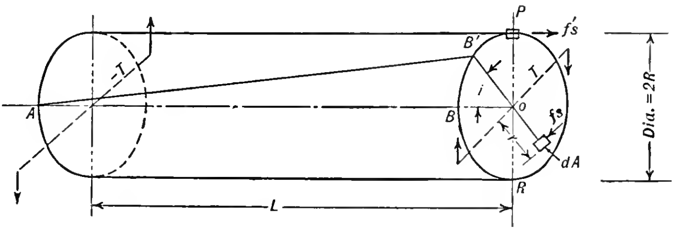


FIG. 6 : 1

other point of area dA a distance r from O, the strain, and consequently the stress, would be in the ratio of r to R, so that $f_s = \frac{r}{R}f_s' = \frac{Gri}{L}$. The moment, dM, due to the stress, f_s, acting on area dA at a distance, r, from the axis of the rod is then,

$$dM = f_s r\, dA = \frac{Gr^2i}{L}\, dA$$

and the moment on the whole area of the section is

$$M = \int \frac{Gr^2i}{L}\, dA$$

which is equal to the applied couple T if the conditions of equilibrium hold. Then, since $\frac{Gi}{L}$ is a constant for any given case,

$$M = T = \frac{Gi}{L}\int r^2\, dA = \frac{GiI_p}{L}$$

where I_p is the polar moment of inertia of the section, $\int r^2\, dA$. But,

$$\frac{Gi}{L} = \frac{f_s}{r} = \frac{T}{I_p}$$

Hence

$$f_s = \frac{Tr}{I_p}$$

Obviously f_s will be a maximum at the surface where $r = R$, so

$$f_{s\,max} = f_s' = \frac{TR}{I_p} \qquad 6:1$$

The polar moment of inertia for any cross-section is equal to the sum of the moments of inertia of the section about its two principal axes. In the case of a circular member the polar moment of inertia is twice that taken about any diameter as an axis. The shearing modulus of elasticity, G, for homogeneous materials such as steel is approximately 0.4 the tension modulus of elasticity, E.

It is obvious that a hollow circular section, a tube, is susceptible to the same method of analysis as that used above on the solid section. The resultant expression for the stress will be identical in form with Formula 6 : 1. The polar moment of inertia for the tube, however, is different from that for a solid section of the same diameter as would be expected, so the stresses will have different values.

6 : 2. Angle of Torsional Deflection — The angle of twist, i, for a circular bar of length, L, may be determined from the above equations since we have,

$$T = \frac{GiI_p}{L}$$

hence

$$i = \frac{TL}{GI_p} \qquad 6:2$$

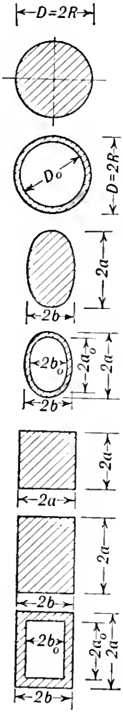
6 : 3. Torsion in Bars Having other Shapes of Cross-Section — Fig. 6 : 2 gives formulas for determining the fiber stresses and deflection angles on members of various shapes. These formulas often prove useful in investigating the stresses in aileron spars and similar members to ascertain whether or not they are "reasonable." They cannot be used for general design since few, if any, data exist as to the allowable stresses on members having such shapes. Until such time as satisfactory allowable stress data become available the strength of all torsional members having shapes other than circular must be determined by test. The formulas given here will, however, be of assistance in developing the trial sections.[1]

6 : 4. Allowable Torsional Stresses in Metal Tubes — Fig. 6 : 3 shows the allowable stresses in torsional shear for mild steel, chrome-

[1] For further data on torsion, see: Chap. X, Vol. II, Fuller and Johnson's "Applied Mechanics;" Chap. VIII, "Strength of Materials," by G. F. Swain; pages 342–401, "Elastizität und Festigkeit," by C. Bach; and "The Circular Arc Bow Girder," by Gibson and Ritchie.

molybdenum steel not heat-treated, nickel steel with various heat-treatments, and aluminum alloy tubes. The allowable stress varies with the ratio of outside diameter to wall thickness. The curves for the mild steel and aluminum alloy tubes have been obtained from fairly comprehensive test data but those for the alloy steels are based on rather meager data and should be used with discretion.

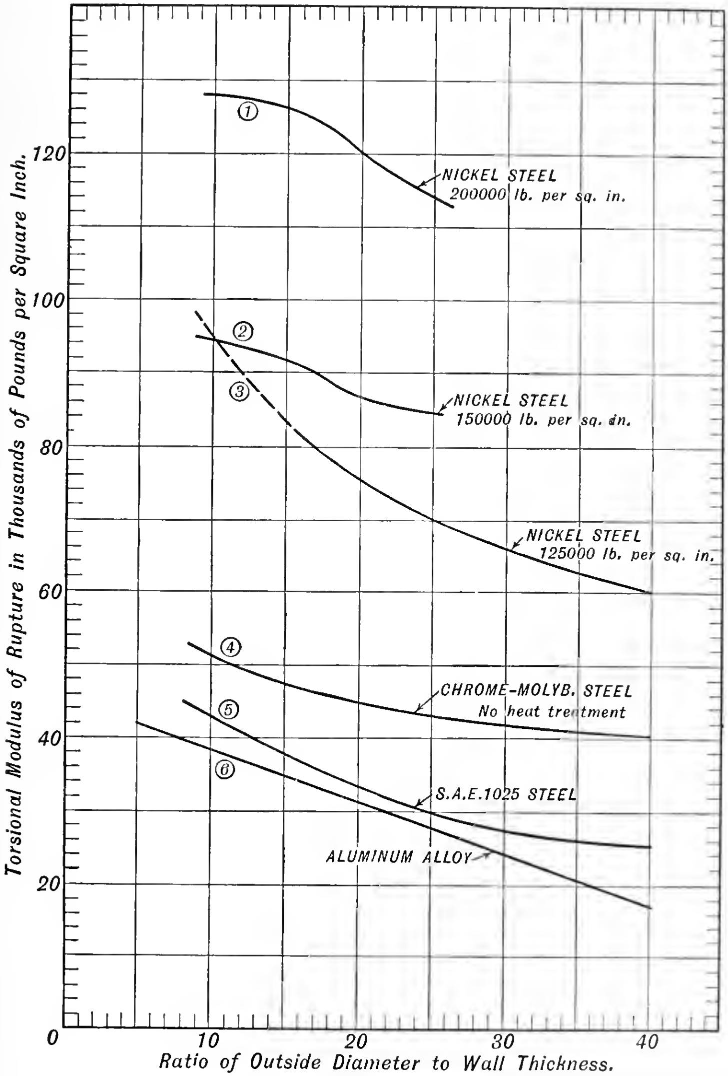
FORMULAS FOR TORSION ON SYMMETRICAL SECTIONS



STRESS DUE TO TORQUE, T	ANGLE OF ROTATION i IN RADIANS
$f_s = \frac{TR}{I_p} = \frac{16\,T}{\pi D^3}$	$i = \frac{TL}{GI_p} = \frac{32\,TL}{G\pi D^4}$
$f_s = \frac{TR}{I_p} = \frac{16\,TD}{\pi(D^4 - D_0^4)}$	$i = \frac{TL}{GI_p} = \frac{32\,TL}{G\pi(D^4 - D_0^4)}$
$f_s = \frac{2\,T}{\pi ab^2}$ at ends of short diameter when $a > b$.	$i = \frac{TL(a^2 + b^2)}{G\pi a^3 b^3} = \frac{4\,\pi^2 I_p TL}{A^4 G}$
$f_s = \frac{2\,bT}{\pi(ab^3 - a_0 b_0^3)}$ at ends of short diameter when $a > b$.	$i = \frac{TL(a^2 + b^2)}{G\pi a^3 b^3(1 - m^4)}$ $a_0 : a = b_0 : b = m.$
$f_s = \frac{3\,T}{5\,a^3}$ (Approx.)	$i = \frac{1}{2.25\,a^4}\frac{TL}{G}$ (Approx.)
$f_s = \frac{T(15\,a + 9\,b)}{40\,a^2 b^2}$ (Approx.) at mid point of long side	$i = \frac{40\,I_p}{A^4}\frac{TL}{G}$ (Approx.)
$f_s = \frac{9\,Tb}{16\,(b^3 a - b_0^3 a_0)}$ at mid point of long side when $a > b$ and $a_0 : a = b_0 : b$	

FIG. 6 : 2

6 : 5. Torsion in Combination with Bending — Pure torsion seldom occurs in members in an airplane structure as the loads are generally so applied that they produce bending in the member as well as torsion. Hence a point such as P in Fig. 6 : 1 in any cross-section on a circular member is subjected to a stress f_b due to bending which acts at right angles to the plane of the cross-section and a shearing stress f_s due to torsion, which acts in the plane of the cross-section and perpendicular to a radius drawn through the point P. Assuming that the bending moment acts in the plane of the diameter PR and the axis of the tube,



Curves 1 and 2 are from Thesis by J. C. Leslie, Mass. Inst. of Tech., 1928
Curve 3 is from U. S. Naval Aircraft Factory Serial Report No. R — 6063
Curves 4, 5 and 6 are redrawn from "Airplane Design"

ALLOWABLE STRESSES ON ROUND TUBES IN TORSION

FIG. 6 : 3

AO, so that the stress due to bending will be a maximum at P, we have $f_b = \frac{M_b r}{I}$ acting normal to the plane of the cross-section and a stress $f_s = \frac{Tr}{I_p}$ acting in the plane of the section. From Mechanics[1] we know that the principal stresses at P under these conditions will lie on oblique planes and their magnitudes will be

$$n_1 = \frac{f_b}{2} + \sqrt{f_s^2 + \left(\frac{f_b}{2}\right)^2}$$

$$n_2 = \frac{f_b}{2} - \sqrt{f_s^2 + \left(\frac{f_b}{2}\right)^2}$$

The greatest intensity of shearing stress will occur on planes at an angle of 45 degrees with the planes of the principal stress, giving

$$f_{s\,\max} = \frac{n_1 - n_2}{2} = \sqrt{f_s^2 + \left(\frac{f_b}{2}\right)^2}$$

When investigating a section to determine the maximum principal stresses it should be observed that the maximum normal stress due to bending occurs at the extreme fiber where the shear due to bending (longitudinal shear) is zero. Similarly, the maximum shear due to bending occurs at the neutral axis where the normal stress due to bending is zero. Study of the formulas of the article will show that the principal stresses will have their maxima either at the extreme fiber or the neutral axis and that these formulas do not have to be used to investigate the stresses at these points for cases of bending without torsion. When torsion is involved the normal stress at the extreme fiber due to bending is normally so much larger than the shear due to bending at the neutral axis that the principal stresses of both kinds will have their maxima at the extreme fiber and no other points on the section need be investigated.

The maximum of the principal stresses, n_1, should not exceed the modulus of rupture stress for the member in pure bending and the maximum shearing stress, $f_{s\,\max}$ should not exceed the modulus of rupture in torsion. For tubular members of steel or aluminum alloy the modulus of rupture in bending is assumed to be equal to the ultimate tensile strength of the material. The modulus of rupture in torsion may be found from Fig. 6 : 3.

6 : 6. Torsion Combined with Bending and Direct Stress — In addition to a combination of torsion and bending many members are sub-

[1] See page 51, Vol. II, "Applied Mechanics," by Fuller and Johnson; page 92, "Strength of Materials," by Swain.

jected to axial loads as well. An axle tube used on an airplane chassis equipped with wheel brakes is an excellent example of this type, being subjected to bending from the component of the ground reaction lying in the plane of the wheel, to torsion from the "braking" couple applied at the brake drum and to direct tension or compression developed in the tube due to its being one of the main members in the chassis tripod. The stresses at any point, P, will then consist of a stress $f = \frac{M_b r}{I} \pm \frac{P}{A}$ acting normal to the plane of the cross-section in which point P lies and a stress $f_s = \frac{Tr}{I_p}$ lying in the plane. P, in the $\frac{P}{A}$ term, represents the axial load on the tube whose area is A. The maximum principal stress will then be

$$n_1 = \sqrt{f_s^2 + \left(\frac{f}{2}\right)^2} + \frac{f}{2}$$

and the maximum shearing stress,

$$f_s \max = \sqrt{f_s^2 + \left(\frac{f}{2}\right)^2}$$

There being no data available as to the allowable stress intensities on members under combined bending, direct stress and torsion, it is considered good practice to assume that the tensile yield point of the material represents the maximum allowable value for the principal stress when the axial load in the member causes compressive stresses which are small as compared with those due to bending. When the axial compressive stresses are high the strength of the member must be determined by test. When the axial stress produces tension in the member the maximum allowable value for the principal stress is taken as the ultimate tensile strength of the material. The maximum allowable shearing stress is taken as the modulus of the rupture in torsion as given in Fig. 6 : 3.

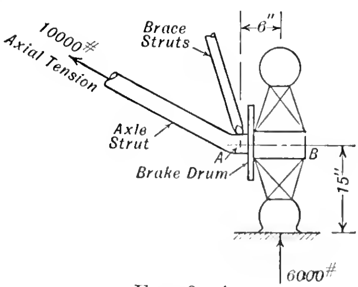


FIG. 6 : 4

Illustrative Example. — An airplane chassis is subjected to a vertical load on each wheel of 6000 lb. as shown in Fig. 6 : 4. It is provided with brakes, the brake drum transmitting torsion directly into the axle which is a 2 1/4-in. — 0.120 nickel steel tube heat-treated to an ultimate tensile strength of 125,000 lb. per sq. in. The coefficient of friction between tires and ground is 0.50 and the tension in the axle

is 10,000 lb. What is the torsional moment on the axle? What are the magnitudes of the bending moment, the maximum shear and maximum principal stresses at A? The area of the axle tube is 0.803 sq. in., its moment of inertia, $I = 0.457$ in.^{4} and its yield point 110,000 lb. per sq. in. Is it safe?

$$\text{Torsional moment at } A = 0.50 \times 6000 \times 15 = 45{,}000 \text{ in.-lb.}$$

$$\text{Bending moment at } A = 6000 \times 6 = 36{,}000 \text{ in.-lb.}$$

$$f_s = \frac{Tr}{I_p} = \frac{45{,}000 \times 1.125}{2 \times 0.457} = 55{,}500 \text{ lb. per sq. in.}$$

$$f_b = \frac{M_b r}{I} = \frac{36{,}000 \times 1.125}{0.457} = 88{,}500 \text{ lb. per sq. in.}$$

$$\frac{P}{A} = \frac{10{,}000}{0.803} = 12{,}500 \text{ lb. per sq. in.}$$

$$f = f_b + \frac{P}{A} = 101{,}000 \text{ lb. per sq. in.}$$

$$f_{s\,\max} = \sqrt{f_s^2 + \left(\frac{f}{2}\right)^2} = 75{,}100 \text{ lb. per sq. in.}$$

$$n_1 = \frac{f}{2} + \sqrt{f_s^2 + \left(\frac{f}{2}\right)^2} = 50{,}500 + 75{,}100 = 125{,}600 \text{ lb.}$$

per sq. in. with the allowable value equal to 125,000.

For this tube the $\frac{D}{t}$ ratio is $\frac{2.25}{0.120} = 18.8$. The tube is heat-treated to 125,000 lb. per sq. in. and, as shown by Fig. 6 : 3, has a torsional modulus of rupture of 76,500 lb. per sq. in. The axle, therefore, has just about sufficient strength in torsion and bending to carry the stresses occurring at section A.

Whenever a couple is to be resisted by forces which do not lie in the plane of the original couple it will be found that, for equilibrium, the forces involved will form two couples the resultant of which will have the same magnitude and lie in the same plane as the original couple. Hence, if the axle tube in Fig. 6 : 4 had been bent at A so that C lay behind, or in front of a vertical plane through AB, the bending moment of 36,000 in.-lb. at A would have been resisted by two moments in tube AC, one of which would have produced torsion, the other simple bending in AC. Similarly, the torsional moment of 45,000 in.-lb. would have been resisted by torsion and bending in tube AC. The determination of the magnitudes of these component moments is most readily obtained graphically as described in Art. 8 : 11 of Chapter VIII.

PROBLEM

6 : 1. An elevator spar is a continuous tube 144 in. long and is arranged so that loads and supports are symmetrical about the center-line of the airplane, the inner supports being 3 in. from the center-line, the outer being 45 in. The masts to which the control wires are fastened are 9 in. from the center-line. A uniformly distributed load of 2.4 lb. per in. extends from each mast outboard to a point 20 in. beyond the outer points of support. This load is so distributed along the chord of the elevators that its center of pressure lies $7\frac{1}{2}$ in. behind the center-line of the spar. Assuming the spar to be a $1\frac{1}{2}$ in. by 0.035 steel tube, S. A. E. 1025, determine the maximum stresses in compression and shear, and the maximum combined stresses, on the tube. Is it safe?

CHAPTER VII

TRUSS ANALYSIS

The preceding chapters have been devoted to the investigation of structures carrying external loads which produced stresses of different intensities on different cross-sections of the members to which the loads were applied. It was found that a given member was subjected to shears and bending moments of varying magnitudes when the lines of action of the external loads were normal to the axis of the member and when all of the loads lay in a plane containing the axis. When the external loads intersected the axis at any angle other than a right angle the member was subjected to axial stresses as well. When the axis of the member did not lie in the plane of the loads the member was subjected to torsional stresses in addition to those due to shear, bending or direct stress. Consideration will now be given to a type of structure in which all of the internal forces produced by the external loads act along the axes of the members of which the structure is composed and hence are constant on all cross-sections throughout the length of the members. Such a structure, which is called an ideal truss, has the following characteristics:

1. It is composed of straight, weightless, co-planar members joined at their ends.
2. The axes of all members meeting at any joint have a common point of intersection at which they are connected by a frictionless pin.
3. The external loads are applied to the truss only at the joints so that the applied loads all lie in the plane of the truss and produce no bending or torsion in the members.

In the practical truss the stresses obtained in the members under the assumption that these conditions are fulfilled are called the "primary" or "main" stresses. Those resulting from the non-fulfillment of any of the above conditions are called "secondary" stresses although they are often equal to and may be even more severe than the "primary" stresses. In the trusses used in airplanes, welded, riveted or bolted connections which are by no means the equivalent of a "frictionless" pin are commonly used and the loads are not always applied to the trusses at the joints. In any case, however, the "primary" or axial stresses in the members are first determined under the assumption

that the given truss conforms to the ideal conditions, such variations as occur in practice being taken care of later in the investigation of the particular members affected.

7 : 1. Statically Determinate Trusses — Trusses may be considered as rigid bodies between supports and hence may be statically determinate, or indeterminate, as regards the outer forces — the reactions — as was shown in Chapter III. They may also be statically determinate or indeterminate as regards the inner forces — the stresses in the members. If a section is cut through a truss and the part to the left of the section is considered as a "free body" it will be in equilibrium under the external loads acting on it and external forces equal to the stresses in the members cut by the section. If it is possible to determine these stresses by the equations of statics the truss is statically determinate with respect to the stresses in the members cut by the section. In general, a truss is statically determinate if each member may be cut by a section which passes through only one or two other members. If it is not possible to determine the internal stresses by the equations of statics alone the truss is statically indeterminate and a complete solution will involve the elastic properties of the members or, in other words, will depend on the deformations of the various parts of the truss. Statically indeterminate trusses will not be considered in this chapter.

Since it is necessary that all parts of a structure must be in equilibrium if the structure as a whole is to be in equilibrium, it follows that if any joint is isolated by passing a section through all of the members meeting at the joint and if the stresses in the members cut are replaced by forces acting at the joint, then these forces must be in equilibrium. Hence the sum of the vertical and horizontal components of the forces at any joint must equal zero, so that each joint in the structure will contribute two independent equations, $\Sigma H = 0$ and $\Sigma V = 0$, toward the number necessary for the determination of the stresses in the members of the truss. Since the axes of all the members at any joint intersect at a common point the condition that $\Sigma M = 0$ does not provide an independent equation. In a truss having n joints there will therefore be $2n$ equations. But the reactions, being unknown external forces, will be applied at two of the joints and will require three of the equations for their determination, leaving $2n - 3$ equations for use in determining the stresses in the members. If b equals the number of members it follows that $b = 2n - 3$ in a statically determinate structure.

All structures which satisfy the above criterion, however, are not necessarily statically determinate throughout since it is obviously possible to remove a member from one part of the structure, making that part non-rigid or unstable, and add it to another part, making that

statically indeterminate. In such cases the criterion should be applied to parts of the truss and if $b < 2n - 3$ for any part that part is unstable, if $b > 2n - 3$ for any part it is statically indeterminate.

7 : 2. Method of Joints — Statically determinate trusses may be investigated by graphical or analytical methods. The analytical methods, of which there are three, will be considered first.

The method of joints is the most general of the methods of truss analysis. It is based on the fact that in a structure which is in equilibrium as a whole, each part, in this case each joint, must be in equilibrium. Each joint may therefore be isolated and treated as a free body acted upon by the external loads applied at the joint and by forces equal to the stresses in the members which are cut when the joint is isolated. If the equations of statics are applied, ΣH and ΣV must equal zero. $\Sigma M = 0$ does not furnish an independent condition. Hence two unknown forces, and only two, may be determined at any joint, unless some special condition exists from which other independent equations may be obtained. It follows, then, that the unknown stresses in two members may be determined at each joint so that, by starting at a joint where there are but two unknowns and taking each joint progressively throughout the structure, it is possible to determine the loads in all of the members. The application of the method is best demonstrated by an example.

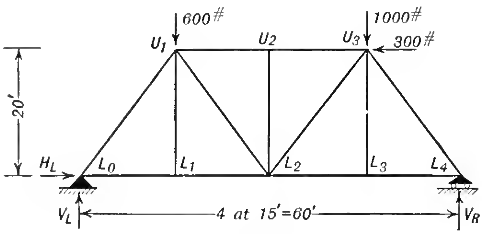


FIG. 7 : 1

The truss shown in Fig. 7 : 1 is statically determined as regards the outer forces since there are but three unknowns, the magnitude and direction of R_L, the magnitude of R_R. There are 13 bars and 8 joints, so $b = 2n - 3$. The truss therefore satisfies the criteria and is statically determinate as regards both outer and inner forces. The reactions are found as follows:

$$-60\,R_R + 600 \times 15 + 1000 \times 45 - 300 \times 20 = 0$$
$$R_R = V_R = 800 \text{ lb. acting upward}$$
$$V_L = 1000 + 600 - 800 = 800 \text{ lb.}$$
$$H_L = 300 \text{ lb. acting toward the right}$$

Let joint L_0 be isolated from the truss since there are but two bars, hence two unknown forces, acting at that joint. It is acted upon by the components, H_L and V_L of the reaction and by F_1 and F_2 representing the unknown loads in L_0U_1 and L_0L_1 as shown in Fig. 7 : 2.

While it is obvious that the forces, as shown, are not in equilibrium since all have components upward or to the right, it is common practice to assume that the unknown forces in the bars act away from the joint. This corresponds with having tension in the members. Then, if the convention for signs is as shown,

$$\Sigma H = 0 = H_L + H_{F_1} + H_{F_2}$$
$$\Sigma V = 0 = V_L + V_{F_1} + V_{F_2}$$

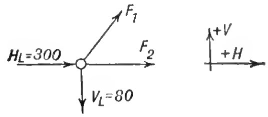


FIG. 7 : 2

where H_{F_1}, H_{F_2}, V_{F_1}, and V_{F_2} are the respective horizontal and vertical components of F_1 and F_2. But a component of the stress in a member, taken parallel to any axis, is to the stress in the member as the length of the member projected on that axis is to the actual length of the member. Then,

$$H_{F_1} = \frac{15\,F_1}{25}, \quad V_{F_1} = \frac{20\,F_1}{25}, \quad H_{F_2} = \frac{30\,F_2}{30}, \quad V_{F_2} = 0$$

and

$$\Sigma H = 0 = 300 + 0.6\,F_1 + F_2$$
$$\Sigma V = 0 = 800 + 0.8\,F_1 + 0$$

Whence,

$$F_1 = -\frac{800}{0.8} = -1000 \text{ lb.}$$
$$F_2 = -300 + 600 = +300 \text{ lb.}$$

The minus sign indicates that F_1 acts in a direction opposite to that assumed, or toward the joint. Member L_0U_1 is therefore in compression. The direction of F_2 was assumed correctly, indicating tension in member L_0L_1. The convention for designating the character of the stress commonly used in aeronautical structures is based on the change in length of the member in question. If it is in tension its length increases, hence a plus sign denotes tension; the length of a compression member is reduced, hence a minus sign denotes compression. By using these conventions and assuming that the unknown forces in the members act away from the joint under investigation the character of the stress will be indicated directly by the sign of the term giving its magnitude.

If joint L_1 is isolated,

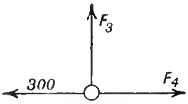
$$\Sigma H = 0 = -300 + F_4$$
$$\Sigma V = 0 = 0 + F_3$$

Hence, $F_4 = +300$ lb. in L_1L_2 and $F_3 = 0$ in L_1U_1. From similar computations the stresses in U_2L_2 and U_3L_3 are zero.

Isolating joint U_1 and omitting bar L_1U_1 which carries no stress:

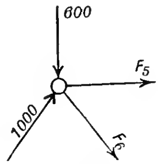
$$\Sigma H = 0 = \tfrac{15}{25} \times 1000 + F_5 + \tfrac{15}{25} F_6$$
$$\Sigma V = 0 = \tfrac{20}{25} \times 1000 - 600 - \tfrac{20}{25} F_6$$

From these equations, $F_6 = +250$ lb. and $F_5 = -750$ lb. Then both F_5 and F_6 act toward the joint and members U_1U_2 and U_1L_2 are in compression. Joint L_2 could be taken next, then U_3 and lastly L_4, and the stresses in all of the members would be completely determined for the conditions and loading shown.



Joint L_1

FIG. 7 : 3



Joint U_1

FIG. 7 : 4

The method of joints is applicable to any statically determinate truss, but its use involves unnecessary labor in many cases, particularly when the stresses in only a few of the members near the center of the truss are desired. In such cases it is best to apply the method of moments or the method of shears, depending on which is easier to apply to the case in hand.

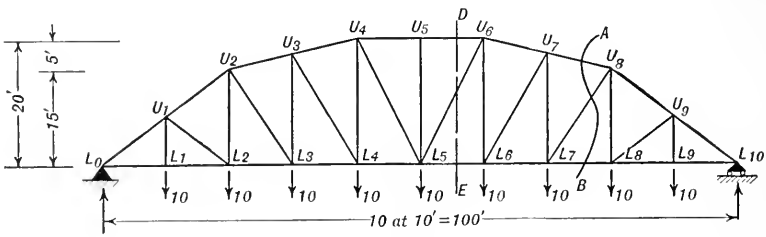


FIG. 7 : 5

7 : 3. Method of Moments — The section AB passed through the truss shown in Fig. 7 : 5 cuts three members, U_7U_8, L_7U_8, and L_7L_8. If the part to the right of the section is isolated from that to the left, and the stresses in the members cut by section AB are replaced by forces F_1, F_2 and F_3 the isolated portion will be acted upon by forces as shown in Fig. 7 : 6. The loads are in 1000-lb. units.

There are then three unknown forces, each assumed to produce tension in the member it replaces, whose magnitudes may be determined from the three equations of statics since the unknown forces must be of such magnitudes that the isolated portion of the structure will be

in equilibrium. By a suitable choice of the origin of moments it is possible to eliminate any two of the forces and evaluate the third from the single condition that $\Sigma M = 0$ about the origin. Since a force will produce no moment about an origin at any point along its line of action, the obvious method for eliminating two of the unknown forces from the equation $\Sigma M = 0$ is to choose the origin at the point of intersection of the lines of action of the forces to be eliminated. For instance, if it were desired to find the magnitude of F_2 in Fig. 7 : 6, it would be necessary to take the origin at the point of intersection of F_1 and F_3 in order to eliminate them as unknowns. The origin, or moment center as it is commonly called, would then be at C, 40 ft. horizontally to the right of joint L_{10}. Then $\Sigma M = 0 = 45 \times 40 - 10(50 + 60) - F_2 \times 58.3$, giving $F_2 = 12{,}000$ lb.

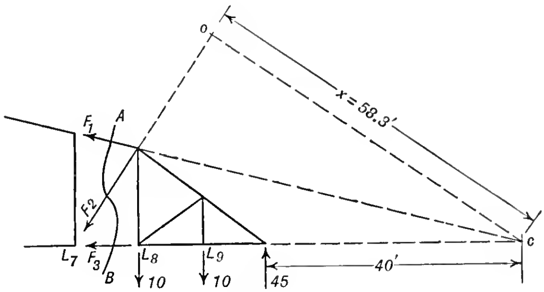


FIG. 7 : 6

Had the stress in L_7L_8 been desired it would have been necessary to eliminate F_1 and F_2 and determine F_3. The moment center would therefore have been at joint U_8, where F_1 and F_2 intersect, and F_3 would be found from

$$10 \times 10 - 45 \times 20 + F_3 \times 15 = 0$$
$$F_3 = +53{,}300 \text{ lb.}$$

The determination of F_3 requires less arithmetical labor than is the case for F_1 or F_2 since the arm of the force about the moment center may be determined by inspection. In finding F_2 the arm was obtained by noting that the angle CL_7O was common to the two right triangles COL_7 and $U_8L_7L_8$, whence $x : 70 = 15 : \sqrt{10^2 + 15^2}$. This computation could have been avoided by passing section AB through the members an infinitesimal distance to the right of U_7L_7 instead of near the middle of the panel. Had this been done the horizontal component of F_2 would have passed through the moment center, C, and the vertical component of F_2 could have been obtained directly, giving,

$$45 \times 40 - 10(50 + 60) - V_{F_2} \times 70 = 0$$

But $$V_{F_2} = \frac{15}{\sqrt{15^2 + 10^2}} F_2$$

Hence, $$\tfrac{15}{18} F_2 = \tfrac{700}{70} \quad \text{and} \quad F_2 = 12{,}000 \text{ lb.}$$

The second method of cutting the section is of more value in the analysis of a given truss than is the first since it gives directly one of the components of the stress in a member. The other component and the stress in the member may be readily obtained and the components are then available for use with the method of joints, if that method offers advantages in the determination of the stresses in adjacent members. It reduces to its simplest form when two of the unknown forces are parallel so that the arm to either of the parallel forces is equal to the distance between them. This is the case of the parallel chord truss where the arm is equal to the depth of the truss and the force, being equal to its component, may be obtained directly.

But in such a case the third unknown force cannot be determined by the method of moments since the moment center would be at infinity, i.e., at the intersection of the parallel chords. The method of moments therefore must be replaced by some other method so that the stresses in the web members of a truss having parallel chords may be obtained. For this purpose we have the method of shears.

7 : 4. Method of Shears — The vertical section, DE, in Fig. 7 : 5 cuts through members U_5U_6, L_5L_6 and L_5U_6. Isolating the part to the right of DE and replacing the forces in the bars cut by F_4, F_5 and F_6 give the structure shown in Fig. 7 : 7.

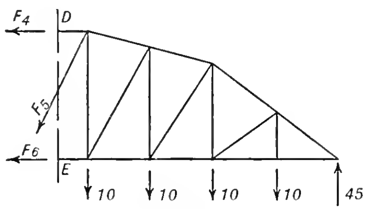


Fig. 7 : 7

Applying the condition that $\Sigma V = 0$ to the isolated portion of the truss and assuming upward forces to be positive,

$$V_{F_4} - V_{F_5} + V_{F_6} - 40 + 45 = 0$$

But V_{F_4} and $V_{F_6} = 0$ since the length of the vertical projection of each of these bars is zero. Then $V_{F_5} = 45 - 40 = 5$. Hence the vertical component of the stress in the inclined web member L_5U_6 is equal to the vertical shear in panel 5–6 and we have the "Method of Shears." By using an inclined section passing through the two chords and a vertical member, in a parallel chord truss having vertical web members, the same method of shears may be employed to determine the stress in the vertical since, with $\Sigma V = 0$, that stress must be equal to the vertical component of the shear on the section.

This method is applicable to trusses having inclined chords if the vertical components of the stresses in the chord members are known, so that the values corresponding to V_{F_4} and V_{F_6} in the above equation for $\Sigma V = 0$ may be used in the equation. The method of shears may therefore be used in combination with the method of moments

for the complete determination of the axial stresses in all members of a truss.

None of the three methods described has such distinct advantages as to be used to the exclusion of the others but each has advantages in certain cases; and by a judicious use of one, two, or of all three of the methods the analysis of a planar truss may be made in a simple and rapid manner. It should be noted that each of the methods is a method of sections and when applying any of them to a given bar in a truss it is well to consider the location and shape of the sections to be used with all three methods so that the simplest and most accurate method may be utilized.

An example will now be given to show how readily the stresses in all members of a parallel chord truss, such as the internal drag truss in an airplane wing, may be determined.

7 : 5. Illustrative Example — Given the truss and loads shown in Fig. 7 : 8, determine the stresses in all of the members.

Procedure. — Draw the truss to scale and indicate the loads acting on it as shown in Fig. 7 : 8. Compute the lengths of the diagonal

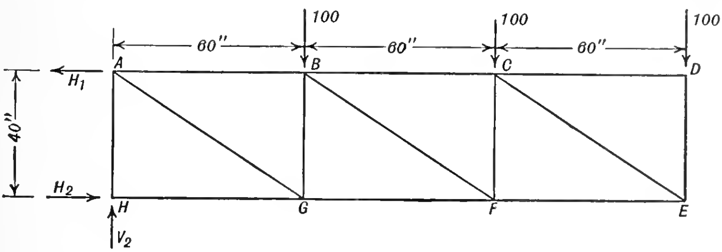


FIG. 7 : 8

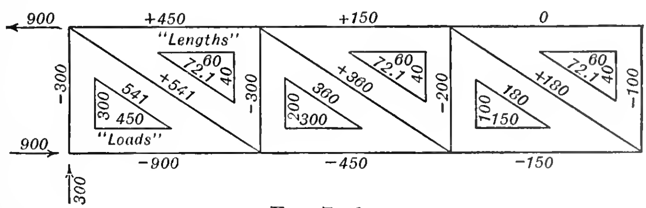


FIG. 7 : 9

members from their horizontal and vertical projections and record the values on the triangles of "lengths" as shown in Fig. 7 : 9. The computation for the length of the hypotenuse of a right triangle may be made on a slide rule by noting that, since $\sec^2 A = \tan^2 A + 1$, $\left(\frac{CE}{FE}\right)^2 = \left(\frac{CF}{FE}\right)^2 + 1$. This gives $CE = FE\sqrt{\left(\frac{CF}{FE}\right)^2 + 1}$. Hence the computa-

tion involves dividing the shorter leg by the longer and taking the square of the result, carefully noting the position of the decimal point. By adding 1.0 to this quantity and resetting the slide rule to obtain the square root of the result the length of the diagonal may be determined by taking the product of the value thus obtained times the length of the longer leg.

The vertical components of the stresses in the diagonals are then obtained by the method of shears and recorded on the vertical sides of the load triangles in each panel. From the ratios of length to vertical projection and of horizontal to vertical projection the stresses and their horizontal components in each of the diagonals may be obtained and recorded on the appropriate sides of the load triangles. The method of joints is then applied, starting at D where there are only two unknowns to be determined. Applying $\Sigma H = 0$ gives $H_{CD} = 0$. Hence the stress in $CD = 0$. Applying $\Sigma V = 0$ gives, $-V_{DE} - 100 = 0$. Then $V_{DE} = -100$ and since DE is vertical the stress is -100. Taking joint E next and applying $\Sigma H = 0$ we have, $-H_{CE} - H_{EF} = 0$. But H_{CE} is the horizontal component of the stress in CE which has been found to be 150. Then $H_{EF} = -H_{CE} = -150$. Isolating joint C next and applying $\Sigma V = 0$ gives, $-100 - V_{CE} - V_{CF} = 0$. But $V_{CE} = -100$, so that $V_{CF} = -100 - 100 = -200$. Applying $\Sigma H = 0$, we obtain, $H_{CD} + H_{CE} - H_{BC} = 0$. Having previously determined that $H_{CD} = 0$ and $H_{CE} = 150$, it follows that $H_{BC} = 150$. If each joint is considered in this way the stresses in each of the members of the truss may be readily determined by simple addition and subtraction, the equations being set up and solved mentally. The complete solution of a parallel chord truss may thus be made with the aid of a slide rule without writing down a single equation and the stresses recorded on the members as shown in Fig. 7 : 9 in a manner that may readily be checked for arithmetical errors.

A partial check of the numerical work may be obtained by computing the reactions and investigating the equilibrium of the joints at which they act. The stress in one of the chord bars may be determined by the method of moments and checked against the value obtained by the methods of joints and shears. In this case the reactions were computed from the applied loads and found to be as shown in Fig. 7 : 9. Then at joint A, $\Sigma H = H_{AB} + H_{AG} - 900 = 450 + 450 - 900 = 0$ and $\Sigma V = -V_{AH} - V_{AG} = 300 - 300 = 0$. At joint H, $\Sigma H = 0 = H_{GH} + 900 = -900 + 900$. $\Sigma V = V_{AH} + 300 = -300 + 300 = 0$. Joints A and H are therefore in equilibrium.

7 : 6. Application to Airplane Wing Frameworks — The wing structure on either side of the center line of the conventional single-bay

biplane is a framework that may be looked upon as being composed of six trusses, corresponding in many respects to the six sides of a box. As shown in Fig. 7 : 10, there are two drag trusses, one in each wing,

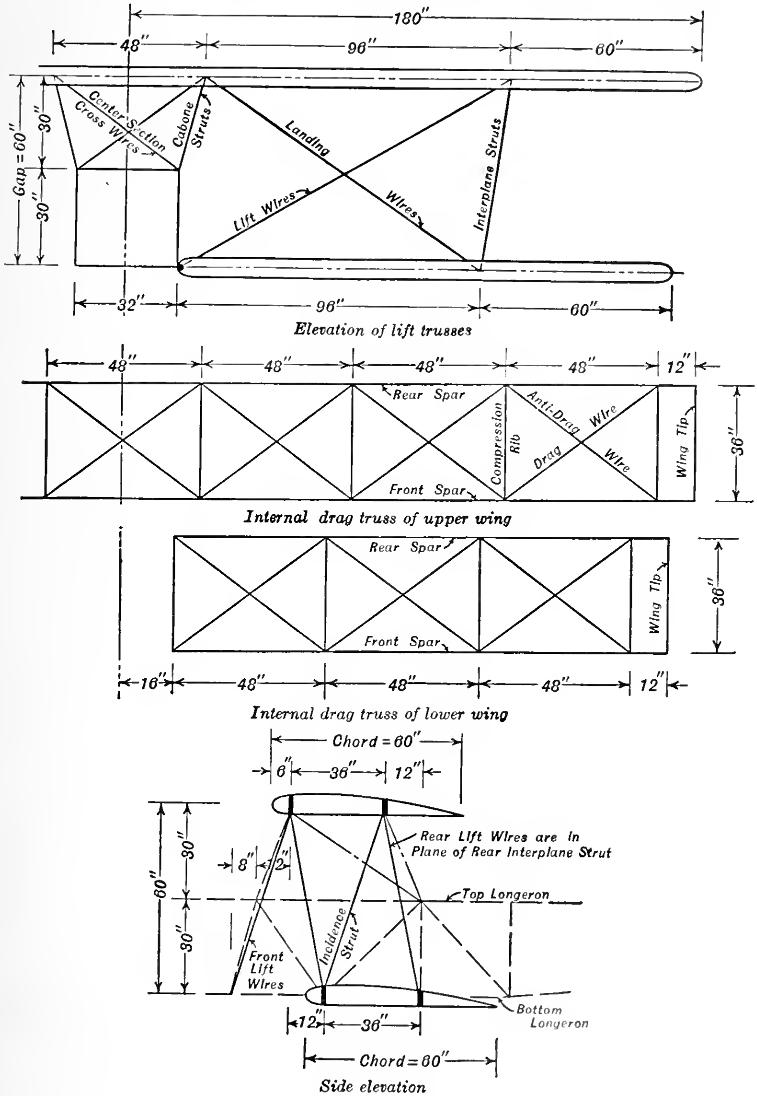


FIG. 7 : 10

made up of the front and rear spars, the compression ribs and the "drag" and "anti-drag" wires. These trusses are inside of the wings and serve to carry the chord components of the loads on the wing

cellule. The front spars of the upper and lower wings taken in conjunction with the front lift wires, the front interplane and cabane struts form the "front lift truss." The rear spars, wires and struts constitute the rear lift truss. The primary purpose of these lift trusses is to carry the lift or "beam" components of the loads. In addition there are two "stagger" or "incidence" trusses, one in the plane of the outer struts and one in the plane of the "center section" or "cabane" struts. The outer incidence truss serves, because of its rigid construction, to equalize the deflection of the front and rear lift trusses. The diagonal strut (or wires) acts to restrain the deflection of the more heavily loaded truss by transmitting some of the load to the more lightly loaded one. This effect, however, can usually be neglected, except in the nose-dive condition, by assuming the diagonal strut or wires lying in the plane of the outer interplane struts to be out of action. Such an assumption is conservative for the conventional structure and renders it statically determinate. The incidence trusses in the plane of the cabane struts transmit the beam and chord components of the load from the upper wings into the fuselage and serve to maintain the position of the upper wing with respect to the rest of the structure.

Although most of the connections between the various trusses forming a wing cellule are made with pinned joints which, for practical purposes, may be assumed frictionless, there are individual members in the trusses which are continuous and so carry bending moments at the panel points. Others may be so connected, as by welding, that bending stresses may be transmitted from one member to another. In the case of the continuous members it is common practice to determine the moments and reactions at the panel points of the truss by the three-moment equation. These reactions are then used as the panel loads for determining the axial loads in the members.

The analysis of the wing cellule shown in Fig. 7 : 10 will now be carried through to determine the axial loads in each of the members of the various trusses, the upper wing spars being considered as continuous, the lower as pin-connected to the fuselage. It will be assumed that the wing spars carry the load imposed upon them by the ribs as though it were uniformly distributed, and transmit it to the panel points of the lift truss. The deflection of the trusses under load will be neglected, the upper wing support points being assumed co-linear. The load will be taken as 1.0 lb. per in. along the span of each spar between the center line of the airplane, or the side of the fuselage in the case of the lower wing, and the outer strut point. The intensity of the load outboard of the outer strut point will be assumed to decrease uniformly to 0.5 lb. per in. at the wing tip. For the spars of the lower

wing this gives a loading curve as shown in Fig. 7 : 11. The reactions are:

$$-96\,R_1 + \frac{1 \times 96^2}{2} + 0.5 \times 60(96 + 30) + \frac{0.5 \times 60}{2}(96 + 20) = 0$$

$$R_1 = 105.5 \text{ lb.} \quad \text{and} \quad R_2 = 35.5 \text{ lb.}$$

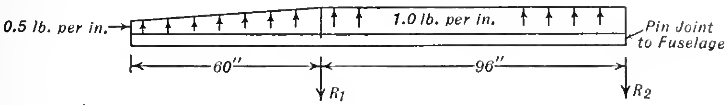


FIG. 7 : 11

For the spars in the upper wing the loading curve is as shown in Fig. 7 : 12. The moment at the outer strut point is

$$M_1 = (0.75 \times 60)\,(0.444 \times 60) = 1200 \text{ in.-lb.}$$

(The centroid of the trapezoid representing the loading on the portion of the wing outboard of the strut point is 44.4 per cent of length of the cantilever tip from the strut point. See Table 3 : 2, Page 45.)

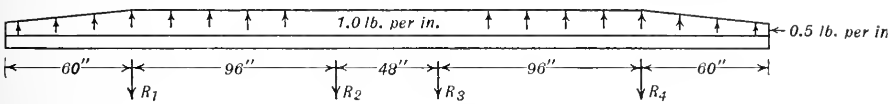


FIG. 7 : 12

Assuming the spars to have constant moments of inertia throughout their lengths and noting that $M_2 = M_3$ by symmetry,

$$1200 \times 96 + 2\,M_2(96 + 48) + M_2 \times 48 = \frac{1 \times 96^3}{4} + \frac{1 \times 48^3}{4}$$

$$M_2 = 398 \text{ in.-lb.}$$
$$R_1 = R_4 = 101.4 \text{ lb.}$$
$$R_2 = R_3 = 63.6 \text{ lb.}$$

We now have the beam components of the panel loads for both lift trusses for a running load of 1.0 lb. per in. If the airplane is considered to be flying in the Low Angle of Attack attitude, the actual loads on the spars for an airplane of this size would be approximately:

Front upper spar.	3.5 lb. per in.
Rear upper spar.	9.0 lb. per in.
Front lower spar.	3.0 lb. per in.
Rear lower spar.	7.7 lb. per in.

The beam components of the panel loads would then be as shown in the following table:

PANEL LOADS

Spar	Loading per in.	Interplane Strut Point	Cabane or Fuselage
Front upper spar.....	1.0	101.4	63.6
	3.5	355	223
Rear upper spar......	1.0	101.4	63.6
	9.0	913	573
Front lower spar.....	1.0	105.5	35.5
	3.0	317	107
Rear lower spar......	1.0	105.5	35.5
	7.7	813	274

The projections and theoretical center line lengths of the members of the lift trusses will now be obtained so that the analysis of the trusses may be completed:

Member	Beam Proj. In.	Chord Proj. In.	Horiz. Proj. In.	B^2	C^2	H^2	L^2	Length In.
Front Lift Wire.........	60	20	104	3600	400	10816	14816	121.9
Rear Lift Wire..........	60	12	104	3600	144	10816	14560	120.7
Front Interplane Strut..	60	12	8	3600	144	64	3808	61.7
Incidence Strut.........	60	24	8	3600	576	64	4240	65.1
Rear Interplane Strut...	60	12	8	3600	144	64	3808	61.7
Front Cabane Strut.....	30	12	8	900	144	64	1108	33.3
Diagonal Cabane Strut..	30	48	8	900	2304	64	3268	57.2
Rear Cabane Strut......	30	12	8	900	144	64	1108	33.3

Assuming the incidence strut to be out of action and considering the front interplane strut point on the lower wing we have, since $\Sigma B = 0$, the beam component in the front strut equals 317 lb. The beam component in the rear strut is 813 lb. The stresses in these struts are then

$$\text{Front strut } 317\left(\frac{61.7}{60}\right) = 326 \text{ lb.}$$

$$\text{Rear strut } 813\left(\frac{61.7}{60}\right) = 836 \text{ lb.}$$

Applying $\Sigma H = 0$ and assuming horizontal components positive to the right gives

$$-H_{\text{strut}} - H_{\text{spar}} = 0 = -317 \times \tfrac{8}{60} - H_{\text{spar}}$$
$$H_{\text{spar}} = \text{stress in lower front spar} = -42.3 \text{ lb.}$$

Isolating the joint at the interplane strut point on the upper front spar and assuming tension in the front lift wire,

$$\Sigma B = 0 = B_{\text{spar}} + B_{\text{strut}} - B_{\text{wire}} = 355 + 317 - B_{\text{wire}}$$
$$B_{\text{wire}} = +672 \text{ lb.}$$

The stress in the front lift wire is then $672 \times \frac{121.9}{60} = 1365$. Applying $\Sigma H = 0$ gives

$$H_{\text{strut}} - H_{\text{wire}} - H_{\text{spar}} = 0$$
$$317 \times \tfrac{8}{60} - 672 \times \tfrac{104}{60} - H_{\text{spar}} = 0$$

H_{spar} = Stress in upper front spar $= 42.3 - 1163 = -1120$ lb. Similarly, the stress in the rear lift wire is $(913 + 813)\frac{120.7}{60} = 3470$ lb. and that in the upper rear spar is -2890 lb. The stress in the rear cabane strut is $573 \times \frac{33.3}{30} = 636$ lb. tension. This produces an added compression in the rear spar between the cabane struts equal to $-573 \times \frac{8}{30} = -150$ lb.

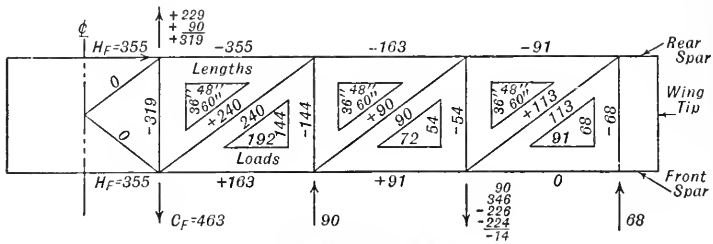
We now have determined the axial loads in the members of the lift truss outboard of the cabane and in the rear cabane strut.

The stresses in the upper wing drag truss due to the chord components of the load on the wings, will next be investigated. It is assumed that the drag trusses in the wings carry the chord components of all loads applied to the wings, hence they must resist the chord components of the load in the interplane struts, the rear cabane strut, and the lift wires. The reactions are furnished by the cabane struts, all of the chord component being carried by the front and diagonal struts with the type of structure shown. The chord component of the load on the wings will be assumed to be uniformly distributed from wing tip to wing tip, and in the Low Angle of Attack condition, to be 15 per cent of the sum of the normal beam components on the two spars. It will also be assumed that the wing spars which form the chord members of the drag trusses act as though they were pin connected at each panel point and that the running load per inch on the wing is concentrated at the nearest panel points. The running load on the upper wing is therefore $0.15(3.5 + 9.0) = 1.88$ lb. per in. and on the lower it is 1.60 lb. per in. These loads act in the drag direction, toward the trailing edge.

The chord components at the upper wing panel points are therefore $48 \times 1.88 = 90$ lb. except at the outer panel point where it is $36 \times 1.88 = 68$ lb. The chord component at the outer strut point due to the front and rear interplane struts is $-317 \times \frac{12}{60} - 813 \times \frac{12}{60} = -226$ lb. forward on the upper wing, backward on the lower. The chord component from the front lift wire is $-672 \times \frac{20}{60} = -224$ lb.; and from the rear lift wire, $1728 \times \frac{12}{60} = 346$ lb. acting aft. The chord component of the rear cabane strut is $573 \times \frac{12}{30} = 229$ lb., also acting toward the rear. The loads on the upper wing drag truss are shown in Fig.

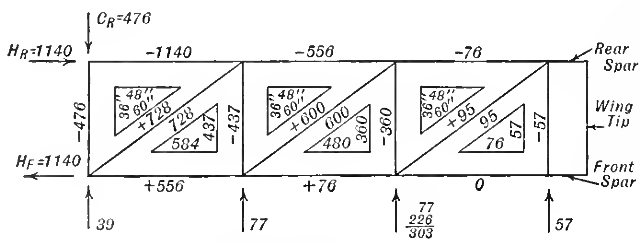
7 : 13. The wires in the center section panel obviously carry no stress since the shear across the panel is 0. The stresses in the members of the upper wing drag truss are also shown in Fig. 7 : 13. Figure 7 : 14 shows the loads and stresses on the lower wing drag truss.

With the beam, chord and horizontal components of the load imposed on the front cabane struts by the upper wing known, it is now possible to determine the stresses in the cabane and the center section of the front spar. The beam component at the front cabane is 223 lb. upward, the chord component 463 lb. in a drag direction and the horizontal



Upper wing drag truss

FIG. 7 : 13



Lower wing drag truss

FIG. 7 : 14

component from the front spar in the outer panel is −1120 lb. due to the action of the lift truss, +355 lb. due to drag. Applying $\Sigma B = \Sigma C = \Sigma H = 0$ at the joint we have:

$$
\begin{aligned}
-B_{\text{front}} - B_{\text{diag}} + 223 \qquad\qquad &= 0 \\
-C_{\text{front}} + C_{\text{diag}} + 463 \qquad\qquad &= 0 \\
-H_{\text{front}} - H_{\text{diag}} + H_{\text{spar}} - 765 &= 0
\end{aligned}
$$

From these equations we find that the stress in the front cabane strut is 455 lb. tension, that in the diagonal strut is 356 lb. compression and that in the front spar, between the cabane struts, is 825 lb. compression.

The primary axial stresses in all of the members of the lift and drag trusses have now been determined and we have found that some of

the members are loaded because of their forming parts of the lift truss, others because of their presence in the drag trusses. The wing spars carry loads from both lift and drag trusses so that the axial load at any section in the spars is equal to the algebraic sum of the loads developed at that section when the spar is considered first as a part of the lift truss then as a part of the drag.

The student should pay particular attention to the method of procedure followed in the solution of this structure in order that it might be analyzed as a series of interdependent planar trusses instead of as a single framework. In general, an analysis based on the resolution of the structure into planar trusses is less laborious than one which considers the structure as a whole.

7 : 7. Space Frameworks — There are some cases of three-dimensional frameworks where it is more desirable to deal with the whole structure as a unit than to break it up into a series of planar elements, and still other cases in which it is necessary to deal with the structure as a unit. The tripod structure used in the modern "bent-axle" type of chassis is a simple example of the latter case.

Such structures will be in equilibrium as regards the external forces if the algebraic sum of the components parallel to any axis of reference of all the forces acting on the structure is zero and if the algebraic sum of the moments of all of the forces about any axis is also zero. From these two conditions we may deduce six equations applicable to the external forces. Three of these apply to the components of the forces parallel to three independent axes: $\Sigma V = 0$, $\Sigma H = 0$, $\Sigma D = 0$; and the other three to the moments of the forces about these axes: $\Sigma M_V = 0$, $\Sigma M_H = 0$, $\Sigma M_D = 0$. In determining the reactions on space frameworks all six of these equations may be utilized, hence the reactions can, in general, be computed if they have six unknown and independent components. With less than six the structure will generally be unstable, with more it will be statically indeterminate with respect to the outer forces.

As regards the inner forces, the stresses in the members, we may obtain three independent equations for the components of the forces in the members at each joint where all members do not lie in one plane, two equations at each joint where the members are all in one plane. If the axes of the members intersect at the center of a frictionless joint, as is assumed to be the case, the equations of moment of the forces in the members about any set of axes will not be independent and will therefore be of no assistance in investigating the joint. Hence, we have three independent equations, and not more than three, at each joint of a space framework at which the members have components

in all three dimensions. Then, if n represents the number of such joints, m the number of joints at which all members lie in one plane, b the number of members in the structure, and r the number of points of support at each of which there may be three unknown reaction components, we may establish a criterion as to whether or not a space framework is statically determinate. That is, for complete determination,

$$b = 3\,n + 2\,m - 3\,r.$$

As in the case of planar structures it is sometimes necessary to establish the direction of some of the reaction components on space frameworks by using slotted holes or similar devices. Assuming that f represents the number of such fixed components our criterion becomes,

$$b = 3\,n + 2\,m - 3\,r + f.$$

Space frameworks, if statically determinate, may be analyzed by successive applications of the method of joints or the method of sections, the methods being modified to take into account the additional components and equations entailed in the use of three coördinate axes. In the case of the tripod it is generally most satisfactory to use the method of joints, isolating the one at which the three members intersect and resolving the stresses in the members into components parallel to the three coördinate axes. Then, by applying the equations of equilibrium, the magnitude of the components may be determined and the stresses in the members obtained. It is sometimes more convenient to pass a section through the tripod, take an axis in the plane of two of the members, and determine the stress in the third by writing the equations of equilibrium for the moment of all forces about that axis. This method is analogous to the method of moments for a planar truss and may, of course, be used where there are more than three members, all but the one being investigated being considered part of the free body isolated by the section.

The following theorems, taken from page 457 of Spofford's "Theory of Structures," are extremely useful in determining the stresses in members of space frameworks:

(a) If several bars of any framework meet at a joint and all but one lie in the same plane, the component normal to this plane of the stress in that bar which does not lie in the plane will equal the algebraic sum of the components normal to the same plane of all the external forces which may be applied at the joint under consideration.

(b) The moment of any force or bar stress acting in a given plane about an axis lying in that plane equals zero.

(c) At any joint at which no outer force is applied, and at which

the stresses in all bars but two have been found to be zero, the stress in each of these bars will also be zero, provided that the two bars do not lie on the same straight line.

When analyzing space frameworks the arithmetical or algebraic work may frequently be very greatly reduced by investigating the structure under the influence of each of the three components of the applied loads rather than under the loads themselves. This is especially true where the load at one joint is to be modified during the course of the investigation, in which case the simplest procedure is to determine the stresses in each member under unit V, H, and D components acting at that joint.

PROBLEMS

7 : 1. Analyze the airplane cellule shown in Fig. 7 : 10 for the same loads as are used in Article 7 : 6. Assume the spars of the lower wing to be vertically below those in the upper and let both front and rear lift wires lie in vertical planes.

7 : 2. The axle and front brace struts of the chassis shown in Fig. 6 : 4 lie in a vertical plane, the upper ends of these struts being respectively 10 and 40 in. to the left of, and 24 in. above point A. The front and rear brace struts lie in a plane, the rear strut being attached to the longeron, which is horizontal, 36 in. behind the front brace strut. Determine the stresses in each strut for unit loads acting vertically upward, horizontally toward the rear and horizontally toward the center-line of the airplane.

CHAPTER VIII

GRAPHICAL METHODS

It has been noted in the foregoing chapters that each of the forces considered has had three characteristics: magnitude, direction and point of application. A force is, therefore, a quantity which may be represented by a straight line, called a vector, the slope and position of which show the line of action and point of application of the force, and the length of which, to some convenient scale, represents its magnitude. The direction in which the force acts along its line of action is indicated by an arrow head. Hence forces may be represented by vectors, and by suitable graphical methods the vectors may be manipulated to solve any statically determinate structure.

In a majority of cases the analytical, or algebraic, treatment of forces will be found to be the more advantageous from the standpoints of speed, accuracy, and facility of presentation in a clear and easily verified manner, but there are many problems in airplane structures to which graphical methods are particularly applicable, the most common example being that of a non-parallel chord truss such as is employed in a fuselage.

8 : 1. Resultants and Components Defined — A force may be defined as any action the effect of which is to change the state of motion of a body. When a single force is applied to a body it tends to accelerate that body in the direction in which the force is acting. If several forces act upon a body each tends to cause motion along its line of action, and the body itself will move in a direction which may or may not coincide with the direction of one of the applied forces. In any case the direction in which the body moves will be a composite, or resultant, direction imposed upon it by the combined action of the system of forces. It is therefore possible to replace the given system of forces by a single force, or a couple, the effect of which would be to produce the same conditions of motion as those induced by the system itself. Such a single force or couple is called the Resultant of the system of forces, and each of the forces of the original system may be called a Component of the resultant.

Attention is called to the fact that the resultant of a system of forces will produce the same acceleration of the body on which it acts and will require the same external forces, or reactions, to hold it in equilib-

rium, as the original system. Its point of application to the body, however, will not be the same as those for the original system, since one point replaces several. Hence, the internal forces or stresses in the body will not be the same as those produced by the original system.

8 : 2. Composition of Forces — If the magnitude and direction of a single force or the resultant of a system of forces is represented by the length and slope of a line, the length of that line projected on any given plane will represent the component of the given force parallel to that plane. Similarly the length of that line projected on any given axis will represent the component of the force parallel to that axis. The discussion which follows will be confined to the effect of forces applied to planar structures, the forces or loads being assumed to act in the plane of the structure.

If the lines of action of two forces intersect they are called concurrent forces, and their resultant may be determined in either of two ways. A force diagram, or force polygon, may be drawn as in Fig. 8 : 1 which shows the two forces AB and BC and their resultant AC in magnitude and direction. Such a diagram may be drawn at any point on the paper so that the forces are not necessarily shown in their true positions. It is not limited to two forces but, as is shown in Fig. 8 : 2, may contain any number of forces such as AB, BC, CD, DE, EF. In constructing a force polygon the forces are drawn in the direction and order in which they act, and the line drawn from the tail end of the vector representing the first force to the head end of the vector representing the last force gives the direction and magnitude of the resultant of the forces. In Fig. 8 : 2 this resultant is AF. A force FA equal and opposite to the resultant and having the same line of action is called the Equilibrant of the system since it closes the force polygon and so satisfies the conditions of equilibrium for the system. That is, with force FA added to the diagram the resultant of all of the forces including FA is zero, hence $\Sigma V = \Sigma H = 0$ for the diagram as a whole.

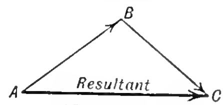


FIG. 8 : 1

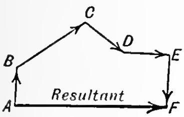


FIG. 8 : 2

Since the force polygon may be drawn in any location the forces are not represented along their true lines of action so that their resultant is not shown in its proper position although its direction and magnitude are correctly indicated. This is apparent from Fig. 8 : 1 where the resultant, AC, does not pass through the point of intersection of the lines of action of the two forces AB and BC, hence it produces a moment about that point and is therefore not the exact equivalent of the original

pair of forces. In order to show the line of action of the resultant in its true position it is necessary to produce the lines of action of the forces until they intersect, then to plot the two forces to scale on the sides of the point of intersection toward which they act as is shown in Fig. 8 : 3. Completing the parallelogram by drawing the lines AD and CD parallel respectively to BC and BA, the diagonal of the parallelogram drawn from the point of intersection of the lines of action of the two forces will establish the resultant BD as to magnitude, direction and line of action. This method is directly applicable only to pairs of concurrent forces and becomes laborious and generally unsatisfactory when applied to a number of forces. It is obvious that it is not applicable to parallel force systems.

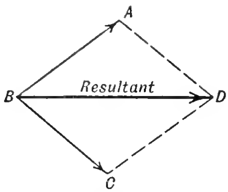
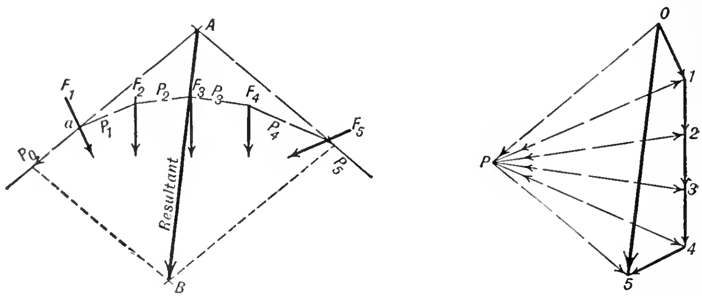


Fig. 8 : 3

It may be extended to apply to any system of forces, however, as shown in Fig. 8 : 4 where it is desired to obtain the resultant of the forces F_1, F_2, F_3, F_4, F_5. First letter the forces from left to right and draw the force polygon as shown in Fig. 8 : 4b. The forces must be laid off in the direction in which they act and in the same order in which they occur in the original system. It is well to letter them so that each end of each force is designated by some figure or symbol.



a Fig. 8 : 4 b

Then $O\ 1$ represents F_1, $1\ 2$ represents F_2, and so on. Now resolve $O\ 1$ into any two components such as $O\ P$ and $P\ 1$, the dash lines in Fig. 8 : 4b, and replace F_1 by these two components. This may be accomplished by drawing the "string" P_0 in Fig. 8 : 4a parallel to the "ray" $O\ P$ in Fig. 8 : 4b until it intersects F_1 at some point such as a. Then from a draw the string P_1 parallel to ray $P\ 1$ until it intersects F_2. Next resolve $1\ 2$ into two components, $1\ P$ and $P\ 2$ which intersect at the same point P as the first two and replace F_2 in Fig. 8 : 4a by these components making the lines of action of the strings representing $P\ 1$ and $1\ P$ coincide. But the component of F_1 represented by the ray $P\ 1$

is equal and opposite to the component of F_2 represented by ray *1 P*. Since the strings representing *P 1* and *1 P* in Fig. 8 : 4*a* were drawn so that they coincide these components neutralize each other and do not affect the original system of forces.

Following the same procedure for the remaining forces we draw the strings representing *2 P*, *P 3*, *3 P*, etc., in Fig. 8 : 4*a*, parallel to the rays *2 P*, *P 3*, *3 P*, etc., in the force diagram, Fig. 8 : 4*b*, so that each of the components is canceled by an equal and opposite component with the exception of the first and last, P_0 and P_5 in this case. These two components may therefore be considered to have replaced the original system of forces and their resultant would therefore be the resultant of that system. Hence, by producing the two strings P_0 and P_5 until they intersect and laying off the components P_0 and P_5 from this point of intersection, completing the parallelogram and drawing in its diagonal, *AB*, we have the resultant of the given forces correctly determined as to magnitude, direction and line of action. A somewhat simpler procedure for this last step is to note that *O 5* in Fig. 8 : 4*b* represents the resultant as to magnitude and direction, and the point *A*, the point of intersection of the strings *O P* and *P 5*, locates one point on the line of action of this resultant. Hence *AB* drawn through *A* and made equal and parallel to *O 5* represents the resultant of the system.

8 : 3. Reactions Obtained Graphically — For equilibrium, the resultant of the reactions on any structure must be equal in magnitude to the resultant of the applied loads. It must also have the same line of

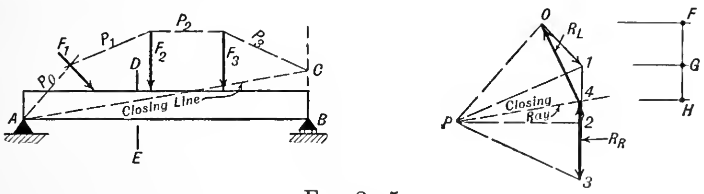


FIG. 8 : 5

action but act in the opposite direction. The method just described for determining the resultant of any system of loads may be extended to cover the determination of the reactions on any structure which is statically determinate with respect to the outer forces. As shown in Fig. 8 : 5 — the force diagram, *O 1*, *1 2*, *2 3* should be constructed, the forces being taken in the order in which they occur. The pole, *P*, is chosen at any point which does not lie in the line *O 3*, and the rays *P O*, *P 1*, *P 2* and *P 3* are drawn in the force diagram to represent the arbitrary components of the forces. The strings P_0, P_1, P_2 and P_3

are then drawn in the space diagram parallel to the corresponding rays. The first string must pass through A, the point of application of the reaction whose direction is unknown, since A is the only known point on the line of action of that reaction. The last string, P_3, should be produced until it intersects the line of action of the reaction whose direction is known, point C in this case. Points A and C are then connected with a string, called a closing line, and a corresponding ray $P\,4$ of indefinite length is drawn in the force polygon. This ray represents one of the components of each of the reactions just as the other rays represent components of the applied loads. Then the rays $3\,P$ and $P\,4$ represent the components of a force acting through C. This force is the right reaction. But since the right reaction is fixed as

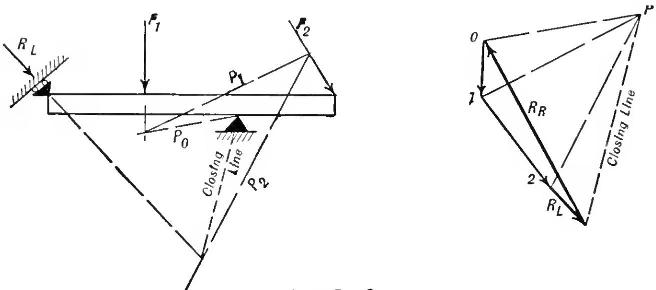


FIG. 8:6

vertical in direction by the type of support shown, the resultant of the components $3\,P$ and $P\,4$, which is $3\,4$ on the force diagram, is vertical and point 4 is therefore vertically above 3 in the force diagram. The line $3\,4$ represents the right reaction both as to magnitude and direction. The resultant of the components $4\,P$ and $P\,O$, which is $4\,O$, represents the left reaction as to magnitude and direction.

The resultant of the reactions, $3\,4$ and $4\,O$, is the same as that of the applied forces F_1, F_2, and F_3 except that it is in the opposite direction. It has components, $3\,P$ and $P\,O$, of the same magnitude as those of the applied forces, hence its line of action must coincide with that of the resultant of the applied forces since it has the same slope and passes through the same point of intersection at A. The conditions of equilibrium are therefore satisfied by the reactions thus obtained.

It is to be emphasized that the first string in the space diagram must pass through the point of application of the reaction whose direction is unknown, since this is the only known point on the line of action of that reaction and hence offers the only definite point at which it is possible to resolve the reaction into the components $4\,P$ and $P\,O$ repre-

sented by the closing line and the first string, or to combine these components to obtain the reaction. It should also be noted that the component $P\,O$ of the left reaction acts in such a way as to cancel the component $O\,P$ of force *1 2*. Similarly, component *3* P of the right reaction cancels P *3* of force *2 3* and the components represented by the closing line, P *4* and *4* P also cancel each other. All of the components represented by the rays and strings thus cancel each other and so produce no effect on the system as a whole.

Figures 8 : 6 and 8 : 7 demonstrate the application of this method to the determination of the reactions on a simple beam having a cantilever overhang, and to a simple truss.

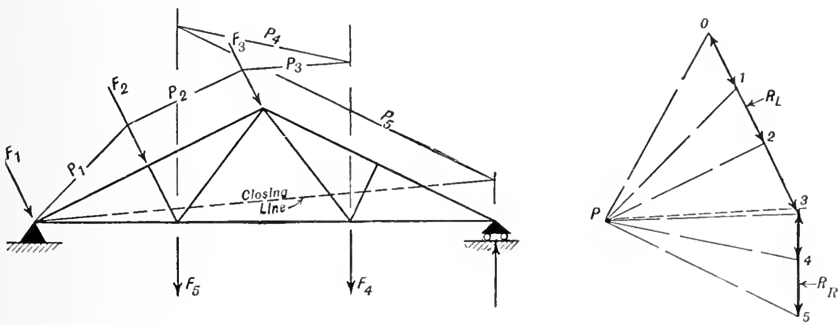


Fig. 8 : 7

8 : 4. Shears Obtained Graphically — The shear at any section of a beam under a series of concentrated loads may be readily obtained by taking the component parallel to the plane of the section of the resultant of all of the forces, including the reaction, to the left, or to the right, of the section. In Fig. 8 : 5 for instance, the shear on plane DE is the vertical component of the resultant of R_L, the left reaction, and F_1. The vertical component of R_L is shown by HF acting upward in the force diagram, and the vertical component of F_1 is FG. Hence the shear is $HF - FG = HG$ acting upward. Since we have considered the forces to the left of the section and have found that the vertical component of their resultant acts upward, the shear is positive. By investigating a number of sections along a given beam in this way it is possible to draw a shear curve for the span just as was done in the case of the analytical method.

8 : 5. Moments Obtained Graphically — The moment of a system of forces about any given point lying in the plane of the forces is equal to the moment of the resultant of those forces about the given point. This may be determined graphically as shown in Fig. 8 : 8 where it is desired to obtain the moment about point a of the forces shown.

The resultant of the given forces is first determined as to magnitude, direction and line of action by one of the methods described above. Then the moment of the forces taken about a as an origin is equal to the resultant, $O\,3$, measured to the scale of forces, times the arm ab measured to the scale of distances.

It should be noted that by prolonging the two strings in the funicular polygon which hold the resultant in equilibrium, $P\,O$ and $P\,3$, and by drawing through a a line parallel to the resultant of the forces the triangle cde is formed. This triangle is similar to the force triangle $P\,O\,3$ since its sides are mutually parallel. Also, Pg and ab are each equal to the altitude of their respective triangles. Then $ab : Pg = de : O\,3$, or $ab \times O\,3 = Pg \times de$. But $ab \times O\,3$ gives the required

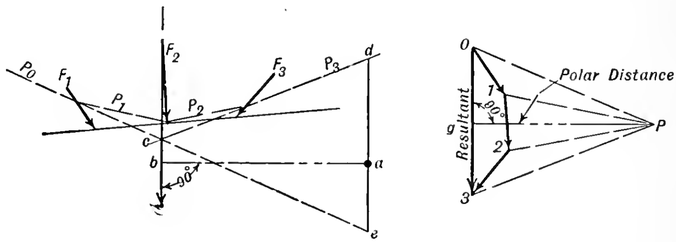


Fig. 8 : 8

moment, hence, $Pg \times de = M$, if Pg in the force diagram is measured to the scale of forces and de in the space diagram is measured to the scale of distances. We thus have the following theorem:

> The moment of any set of co-planar forces about any point in their plane is equal to the product of the length of a line passing through the given point and parallel to the resultant of the forces measured, to the scale of distances, between the strings holding the resultant in equilibrium, times the length of the line, measured to the scale of forces, drawn through the pole of the force diagram perpendicular to the resultant of the forces.

This theorem may be used to determine the bending moment at any point along the axis of a beam as shown in the following example where it is desired to determine the bending moment at section AB of the beam shown in Fig. 8 : 9.

After having found the reactions graphically, it is noted that the bending moment at AB is the moment of the resultant of the left reaction and F_1, these being the forces to the left of the section, about the neutral axis. This resultant is represented by the dash line, $1\,a$, in the force diagram of Fig. 8 : 9 and it is held in equilibrium by the string

P_1 and the closing line. Hence, we draw through C, the point of intersection of the neutral axis of the beam with the section AB, a line RS parallel to the resultant *1 a* and measure, to the scale of distances, the intercept between the strings that hold *1 a* in equilibrium, giving $RS = 7.75$ in. We then draw Pb through the pole of the force polygon normal to the resultant *1 a* and measure its length to the scale of forces, giving 10.5 lb. The bending moment at section AB due to the forces to the left of AB is then $10.5 \times 7.75 = 81.4$ in.-lb. Note that if AB had been taken just to the left of the point of application of F_1 the resultant would have been the left reaction only and the bending moment would have had a different magnitude. By repeating the construction at several sections along the beam the bending moment

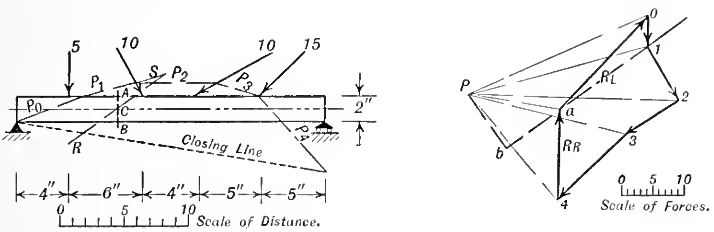


FIG. 8 : 9

at each section may be determined and a curve of moments drawn for the beam.

Attention is called to the fact that when the loads and reactions are all vertical the resultant at any point is vertical and the construction is considerably simplified. In such circumstances the " polar distance " — the length of the normal from the pole to the resultant of the forces acting to either side of the section — is a constant for all sections along the beam. It is therefore convenient to choose a pole at a point such that the polar distance will represent 10, 100, 1000 or other even number of pounds when plotted to the scale of forces. This is illustrated in Fig. 13 : 6, page 283.

It should be noted that the construction just described is not directly applicable to a distributed load. When the bending moment under such a loading is to be obtained it is necessary to divide the load into sections and substitute for the distributed load on each section its resultant, which will be a concentrated load. If the sections taken are short this substitution involves little error and the method described above may be used for the determination of the bending moment at any section.

8 : 6. The Graphical Analysis of Trusses — The analysis of parallel chord trusses is best made by a combination of the methods described

in Chapter VII. In the case of a non-parallel chord structure, such as a fuselage truss, the analysis is generally most readily carried out by graphical methods. The procedure used is analogous to the analytical method of joints. That is, each joint is isolated and the known forces in the bars cut are applied at the joint. The resultant of these known forces and any external load at the joint may then be resolved into two components representing the unknown forces in two members at the given joint. Hence, as in the analytical method, the stresses in two bars may be determined at each joint. This method is most readily explained by applying it to the determination of the stresses in a simple triangular frame as shown in Fig. 8 : 10.

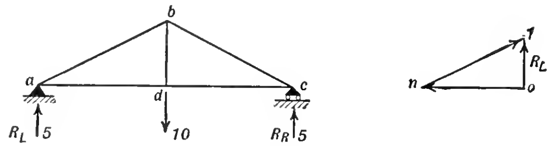


FIG. 8 : 10

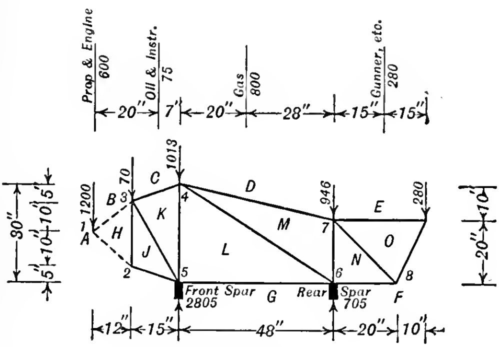
Isolating joint a, the only external force acting is the left reaction, a vertical force of 5. This force is held in equilibrium by the stresses in the bars ab and ad, and the lines of action of these stresses must coincide with the axes of the bars since they are assumed to produce no moment in the bars. Hence, we lay off the force $O\ 1$ equal to R_L and resolve it into two components $n\ 1$ and $O\ n$ parallel to ab and ad respectively, as shown in Fig. 8 : 10. The resultant of the equilibrants of these two components, if their lines of action coincided with the axes of the bars, would act at joint a and would be equal and opposite to R_L. These equilibrants therefore represent the stresses in bars ab and ad since they, and they alone, satisfy the conditions for equilibrium at the joint. Knowing the stress in bar ab, we can isolate joint b next, consider the stress in ab as a force at that joint and determine the stresses in bd and bc, and so on throughout the structure.

Such a method of isolating and investigating each joint is slow and laborious. It involves the use of considerable care in the character of the stress in a member so that the forces and their equilibrants will not be confused. By employing a system of notation, known as Bow's Notation, the analysis of all of the joints in a statically determinate structure may be made on one force diagram without confusion and the character of the stress in any member may be readily obtained.

8 : 7. Bow's Notation — Going around the space diagram of a truss in a clockwise direction letters or other symbols are so placed that

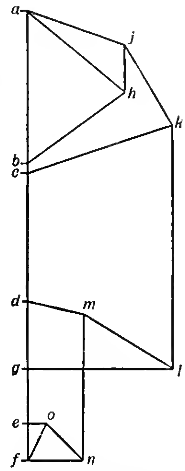
each external force lies between two letters, and only two. Letters or symbols are then placed between the bars in the truss so that each bar lies between two letters, and only two. Each external force or bar stress may then be designated by the two letters between which it lies.

8 : 8. Illustrative Problem — An engine nacelle located on the lower wing of a bombing airplane is extended to carry a machine gunner. The basic loads which are carried by the nacelle structure are shown at the top of Fig. 8 : 11. It is desired to obtain the stresses in the members of one truss assuming that the basic loads are applied to the truss



Loading diagram

FIG. 8 : 11

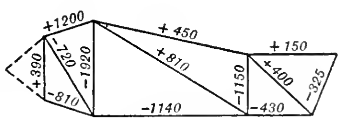


Force and stress diagram

FIG. 8 : 12

at panel points and are divided equally between the trusses on each side of the nacelle. The load factor will be assumed to be four, whence we obtain the panel loads shown in the figure.

The dotted lines represent fictitious members assumed to carry the engine and propeller load back to the members to which the engine ring is actually attached. The reactions are furnished by the front and rear wing spars as shown.



Stress summary

FIG. 8 : 13

We first letter the diagram in a clockwise direction so that each external force, including the reactions which in this case were determined analytically, and each bar will lie between two letters and only two. A force diagram is then drawn and lettered as shown in Fig. 8 : 12, the forces being taken in the same order as they occur in Fig. 8 : 11, viz., AB, BC, CD, DE, EF, FG and GA. Since it is possible to determine the stresses in only two bars at any joint, we shall start at

joint 1, that at which the two fictitious bars intersect. Resolving the force *ab* into two components, *bh* and *ha*, which are parallel to the corresponding bars, we locate point *h* in the stress diagram. Then *ha*, measured to the scale of forces, gives the load in bar *AH* so we may now isolate joint 2 and resolve the load *ah* into two components, *hj* and *ja*, representing the stresses in bars *HJ* and *AJ*. This is accomplished by drawing lines through *h* parallel to *HJ* and through *a* parallel to *AJ*, their point of intersection establishing the point *j* which determines the magnitudes of the stresses in *AJ* and *HJ*. Isolating joint 3 and considering the members in a clockwise order, we observe that we encounter first the bars having the two unknown forces *ck* and *kj*, then the known forces, *jh*, *hb* and *bc* [note that the force diagram for *jh*, *hb* and *bc* is already established by the construction used]. It is now necessary to resolve the resultant of these three forces, *jc*, into the components *ck* and *kj* representing the stresses in the corresponding bars. As before, this is accomplished by drawing through *c* parallel to *CK* and through *j* parallel to *JK*. The point of intersection, *k*, determines the magnitude of the components representing the stresses in these bars. By a continuation of this process, considering the structure joint by joint, we can complete the stress diagram. If no error has been made, the diagram should close, that is, the string representing the last bar, *eo*, should be parallel to bar *EO* and pass through the point *O* obtained as the intersection of *fo* and *no*. A stress diagram which does not close indicates that the drafting has been carelessly done or that the forces used are not in equilibrium.

If the original force diagram, *ab*, *bc*, *cd*, *de*, *ef*, *fg*, *ga* closes it indicates that $\Sigma V = 0$ and $\Sigma H = 0$, hence lack of closure of the stress diagram indicates that $\Sigma M \neq 0$. This is generally the case, if the drafting is good, so it is well to investigate the sum of the moments of the external forces about some point before the stress diagram is started. The time and labor involved in this preliminary check of the force system will usually be less than that required to determine the source of error in and correct a stress diagram which does not close.

8 : 9. Determination of Character of Stress — The stress diagram, Fig. 8 : 12, suffices for the determination of the magnitude and character of the stresses in the members. The magnitudes are obtained directly by scaling the lengths of the lines representing the stresses in the various members to the scale used in laying out the forces. The character of the stress in any bar is obtained by choosing one of the joints at the end of that bar and reading around it in a clockwise direction to determine the designation of the member. Thus, if we choose joint 2 the lower chord of the nacelle truss to the right of 2 is found to be *JA*.

Having chosen the left end of the bar, or the left end of the line of action of the stress, we must be consistent and use the left end of the corresponding component in the stress diagram. Hence, we use point *a* and note that the force *ja* corresponding with bar *JA* acts toward that point. As in the case of the analytical method of joints, a force acting toward a joint indicates compression in the member; one acting away from the joint, tension. Bar *JA* is therefore in compression.

Considering each member in this way, reading around the reference joint in a clockwise direction in each case, we may determine the character of the stress in each member. It is to be noted that it makes no difference which end of the bar is chosen so long as the corresponding end of the force in the stress diagram is used.

Had the forces been considered in a counter-clockwise order when lettering them, it would be necessary to read around each joint in a counter-clockwise direction when determining the character of the stresses. Bow's notation may be used equally well in either direction but it must be used consistently in both space and force diagrams.

8 : 10. The Use of Phantom Members — Where the arrangement of members is such that there are three unknowns to be determined at any joint recourse must be had to the use of auxiliary or "phantom" members to obtain a solution. Such a condition frequently occurs in the side trusses of fuselage structures where a more or less complicated arrangement of the members in a panel is used in order to provide openings for doors or to avoid interference with some item of equipment. The framing shown in Fig. 8 : 14 is typical.

The stress diagram for such a structure may be drawn in exactly the same way as that for the nacelle shown in Fig. 8 : 11 until it becomes necessary to locate point *l*. This point may not be located directly since there are three members at joint 1 having unknown loads and a similar situation exists at joint 2. That is, the component of the known force represented by the ray *ck* in the stress diagram and the load *CD* which is applied at joint 1 must be divided among the three members *KL*, *LN*, and *DN*. As has previously been stated the graphical method permits the determination of the stresses in two members, and only two, at each joint. It is therefore necessary for us to determine the stress in one of the three members, *KL*, *LN* or *DN* if the solution is to be completed. This may be accomplished by replacing the members *LM*, *LN*, and *MN* by the single phantom member *NN'*, shown dashed, and noting that the moment center for the determination of the stress in *DN* is at the same point — the right reaction — whether these three members or the phantom is acting in the panel. This is apparent if a vertical section be assumed through the truss just

to the left of NO and the forces to either side of that section are considered.

Hence, with the phantom NN' we can obtain the true stress in DN by locating point n' at the intersection of rays kn' and gn' and the point n at the intersection of nn' and dn. The stress in DN being thus determined, the phantom member may be replaced by the original bars, whence l will be at the intersection of kl and ln and m should be the common point of intersection of lm, mg and mn. It is obvious that the magnitude of DN could have been determined analytically and the use of the phantom could thus be obviated.

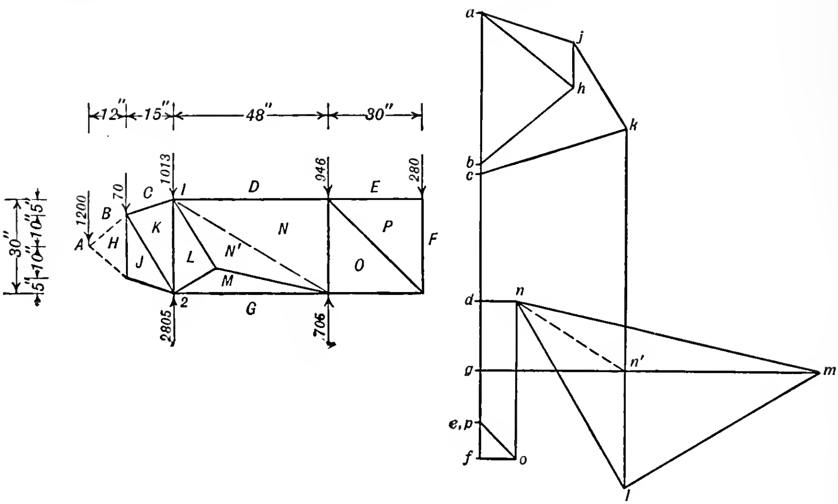


FIG. 8 : 14

When there are more than three unknowns to be determined at any joint the stresses in all but two must be determined analytically because more than one phantom cannot be used in any one panel. Such structures are, however, seldom encountered in practice. The rest of the diagram presents no problems and it may be solved by simple graphics. Attention should be called to the fact that this construction would not have been required had the stress diagram been constructed from the other end of the force diagram, i.e., from the points e and f. While this is usually the case it is not always practicable to construct the diagram from both ends and the use of a phantom member will be found of great assistance.

Another use for a phantom bar occurs where a force is applied at some interior panel point in a structure. In such cases the lettering of the diagram becomes confusing and the simplest solution of the

problem is to move the force along its line of action until it lies outside of the framework. It may then be treated as an external force on the structure and lettered accordingly. The line of action of the force is replaced by a phantom member extending from the new location of the force to its original point of application at the internal panel point, and this phantom member is lettered and used in the stress diagram exactly as though it were an actual member. The load in it will, of course, be equal to the applied force and, as it will have no effect on the stresses in the members it intersects, it may be neglected in the design of the actual members in the structure.

8 : 11. Vectorial Treatment of Moments — A couple may be represented graphically by a straight line drawn normal to the plane of the couple, the length of the line indicating the magnitude of the couple and its direction showing the direction of rotation. The convention usually adopted to indicate direction is to draw the vector acting toward the plane of a clockwise couple and away from the plane of a counterclockwise moment. In this way moments or couples may be combined or resolved by processes identical with those used for forces as shown by the following example.

The front spar of a cantilever monoplane is constructed so that the axes of the two halves of the spar meet at an angle at the center-line of the airplane as shown in Fig. 8 : 15. The bending moment at B due to the portion of the front spar to the left of B is 80,000 in.-lb., the moment on the portion to the right of B being the same if the loads are symmetrical. The load producing this moment in section AB does not lie in the plane of the axis of the spar BC, so pure bending in AB produces both bending and torsion in BC.

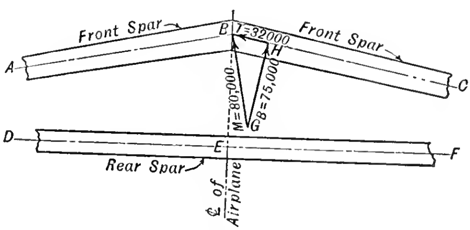


Fig. 8 : 15

In order to determine the magnitude of the moments so produced we will investigate them vectorially by making GB in Fig. 8 : 15 represent a clockwise moment of 80,000 in.-lb. acting at B in a vertical plane passing through AB. Resolving GB into two components, HB and GH, respectively parallel and perpendicular to BC, we find the magnitude of HB to be 32,000 in.-lb., of GH, 75,000 in.-lb. Hence there is a torsional moment of 32,000 in.-lb. tending to rotate spar BC about its axis, as well as a simple flexural moment of 75,000 in.-lb. produced in BC by the moment of 80,000 in.-lb. in AB.

PROBLEMS

8 : 1. Determine the reactions for the truss shown in Fig. 3 : 4 by graphical methods.

8 : 2. Determine the reactions for the spar shown in Fig. 3 : 3 graphically by breaking the uniformly distributed load up into 10-in. segments and applying the resultant of the load on each segment as a concentrated load acting at the center of the segment.

8 : 3. Draw a curve of bending moments for the beam shown in Fig. 3 : 3, using the graphical method for their determination.

8 : 4. Determine the reactions and the stresses in each of the members of the truss shown in Fig. 7 : 1.

CHAPTER IX

DESIGN OF SIMPLE TIES AND COLUMNS

In Chapters VII and VIII methods were described for determining the stresses in an ideal truss. In such a structure the internal stresses act along the axes of the truss members, subjecting them to simple tension or compression. The members carrying tension are often called "ties," and those carrying compression are known as "struts" or "columns." The design of members subjected to simple tension or compression is discussed in this chapter, ties being considered first as their analysis involves fewer factors than does that of columns.

9 : 1. Design of Ties — When a tie is subjected to axial tension it is normally assumed that the stress is uniformly distributed over the cross-section. This assumption is very nearly correct except for local variations in distribution at points where the loads are applied and near points of abrupt change in the cross-sectional dimensions. Using this assumption the strength of the member can be determined by multiplying the ultimate tensile strength of the material used by the net cross-sectional area. The margin of safety can then be determined by comparing the resulting strength with the total load to be carried.

The members most commonly used for ties in airplanes are round, square, and streamline wires or tie-rods and cables. Normally, it is not necessary to go to the trouble of computing net areas and unit stresses for members of these types as standard sizes are available that are rated according to ultimate strengths which have been determined from tensile tests on specimens of the actual wires. The margins of safety can therefore be obtained from a comparison of the loads to be carried and the rated strengths. The rated strengths and some additional data on the more common types of ties are given in Table 9 : 1.

Near the points of connection, where the load is applied to a tie, the distribution of stress over the cross-section will not be uniform, and the member must either be larger at these points or an allowance must be made for "terminal efficiency." In the cases of ties with threaded ends, such as streamline wires and rolled or swaged tie-rods of round or square section, the ends are of greater cross-section, even after the threads are cut, than the central part of the length, so the strength of the latter portion may be considered critical for the member. Cable ends may be spliced so that the full strength of the cable can be devel-

TABLE 9 : 1

STRENGTH OF STANDARD TIES

Round Swaged Tie Rods		Streamline Wire		Steel Cable 19 Strand Non-flexible					Steel Cable 7 × 7 Flexible			Steel Cable 7 × 19 Extra Flexible		
Size and Threads per Inch	Rated Strength Pounds	Size and Threads per Inch	Rated Strength Pounds	Diameter in Inches		Approx. Weight per 100 ft. Pounds	Rated Strength Pounds	E. A. Pounds	Approx. Weight per 100 ft. Pounds	Rated Strength Pounds	E. A. Pounds	Approx. Weight per 100 ft. Pounds	Rated Strength Pounds	E. A. Pounds
6–40	1,000	6–40	1,000	0.031	1/32	0.30	185	20,600						
10–32	1,900	10–32	2,100	0.062	1/16	0.78	500	52,800	0.81	480	48,000			
12–28	2,600			0.078	5/64	1.21	780	87,200*	0.83	550	55,000*			
1/4–28	3,400	1/4–28	3,400	0.094	3/32	1.75	1,100	125,700*	1.45	920	83,900			
		5/16–24	4,600	0.109	7/64	2.60	1,600	176,500						
5/16–24	5,700	5/16–24	6,100	0.125	1/8	3.50	2,100	228,000	2.45	1350	123,100*	2.88	2,000	155,000
3/8–24	8,500	3/8–24	8,500	0.156	5/32	5.50	3,200	359,000*	4.67	2600	271,000	4.44	2,800	255,000
7/16–20	11,500	7/16–20	11,500	0.187	3/16	7.70	4,600	529,000	5.80	3200	333,000*	6.47	4,200	375,000
1/2–20	15,500	1/2–20	15,500	0.218	7/32	10.00	6,100	738,000	8.30	4600	479,000*	9.50	5,600	500,000*
		9/16–18	20,200	0.250	1/4	13.50	8,000	1,080,000*	10.50	5800	604,000*	12.00	7,000	644,000*
		5/8–18	24,700	0.281	9/32							14.56	8,000	814,000
				0.312	5/16	20.65	12,500	1,687,000*	16.70	9200	958,000*	17.71	9,800	1,026,000*
				0.344	11/32							22.53	12,500	1,309,000
				0.375	3/8							26.45	14,400	1,625,000*

From Air Corps Specifications and Transactions A. S. M. E., Vol. 43, pp. 767–788. "A Study of the Elastic Properties of Small-size Wire Cable." By R. R. Moore.

* Computed using estimated values of A.

For least work or deflection computations the areas of swaged or streamline wires may be taken as one one-hundred and fifty-thousandth of the rated strength in pounds.

oped, but the splice commonly employed has an efficiency of about 85 per cent. Even the best forms of terminal for hard round wire will not develop more than about 85 per cent of the tensile strength of the wire; and even when apparently properly made, they may fail at as low as 15 per cent.

In airplane design a tension load is often carried by a pair instead of a single wire. The object of this is to obtain a more reliable structure, so that if one of the pair breaks the structure will not necessarily collapse. Since it is not practicable to rig the airplane in such a way that the total load is divided equally between the two wires of a pair, either wire should be large enough to carry 60 per cent of the total load.

When greater rigidity is desired than can be given by a wire, or where allowance must be made for reversal of stress, a tube, channel, or other structural shape is used for a tension member. When computing the tensile strength of any of these shapes, care must be taken to use the net area of the cross-section. Where the members are connected by welding the net area is equal to the gross area of the section, any weakening effect of the welding being allowed for by the use of a reduced allowable unit stress. Table 9 : 2 gives the tensile strengths at various allowable unit stresses of the more commonly used sizes of round tubes, and also the weights of these sizes of both steel and aluminum alloy tubing. Tables 9 : 3, 9 : 4 and 9 : 5 give the areas, moments of inertia, and section moduli of practically all the sizes of round tubes likely to be encountered in airplane design.

Table 9 : 6 gives similar values for a number of streamline tubes the dimensions of which conform to the Army's Tentative Standard Streamline Section of 1926. Manufacturers of aluminum alloy tubing have drawn sections which deviate from these dimensions to so great an extent that the properties given in the table cannot be used in design. The properties of such sections must be obtained from the producers. The dimensions of the seamless streamline steel tubing drawn by the Summerill Tubing Company of Bridgeport, Pa., differ slightly from those given in Table 9 : 6 but the properties are practically the same. For the Summerill sections

$$\rho = 0.202\,D$$
$$I_{xx} = 0.041\,AD^2$$
$$I_{yy} = 0.178\,AD^2$$

where D is the outside diameter of the basic round tube from which the streamline section is obtained and A is the area of the tube. The center of gravity of these sections lies on the long axis at a point 48 per cent of the length of that axis behind the leading edge.

TABLE 9:2

WEIGHTS AND TENSILE STRENGTHS OF STANDARD ROUND TUBES[1]

Size		Weight per 100 Inch		Tensile Strengths, Pounds per Square Inch						
Diameter, Inches	t	Steel	Aluminum Alloy	36000	55000	60000	80000	100000	125000	150000
3/16	0.035	0.48 lb.	0.17 lb.	600	920	1010	1340	1680	2100	2520
1/4	0.035	0.67	0.24	850	1300	1420	1890	2360	2960	3550
5/16	0.035	0.86	0.31	1100	1680	1830	2440	3050	3810	4580
3/8	0.035	1.06	0.38	1350	2060	2240	2990	3740	4670	5610
1/2	0.035	1.45	0.52	1840	2810	3070	4090	5110	6390	7670
5/8	0.035	1.84	0.66	2340	3570	3890	5190	6490	8110	9730
	0.058	2.93	1.05	3720	5680	6200	8260	10330	12910	15500
3/4	0.035	2.23	0.80	2830	4320	4720	6290	7860	9830	11790
	0.058	3.57	1.28	4540	6930	7570	10090	12610	15760	18910
7/8	0.035	2.62	0.94	3330	5080	5540	7390	9240	11550	13860
	0.058	4.22	1.51	5360	8190	8930	11910	14890	18610	22330
1	0.035	3.01	1.07	3820	5840	6370	8490	10610	13260	15920
	0.049	4.15	1.48	5270	8050	8780	11710	14640	18300	21960
	0.058	4.86	1.74	6180	9440	10300	13730	17160	21460	25750
	0.065	5.41	1.93	6870	10500	11460	15270	19090	23870	28640
1 1/8	0.035	3.40	1.21	4310	6590	7190	9590	11990	14980	17980
	0.049	4.69	1.68	5960	9110	9940	13250	16560	20710	24850
	0.058	5.51	1.97	7000	10690	11670	15550	19440	24300	29160
1 1/4	0.035	3.78	1.35	4810	7350	8020	10690	13360	16700	20040
	0.049	5.24	1.87	6660	10170	11090	14790	18490	23110	27730
	0.058	6.15	2.20	7820	11950	13030	17380	21720	27150	32580
	0.065	6.86	2.45	8710	13310	14520	19360	24200	30250	36300
1 3/8	0.049	5.78	2.07	7350	11230	12250	16330	20410	25510	30620
	0.058	6.80	2.43	8640	13200	14400	19200	24000	30000	36000
1 1/2	0.049	6.33	2.26	8040	12290	13400	17870	22340	27930	33510
	0.058	7.45	2.66	9460	14450	15770	21020	26280	32850	39420
	0.065	8.30	2.97	10550	16120	17580	23440	29300	36630	43950
	0.083	10.47	3.74	13300	20320	25160	33540	41930	52410	62900
1 1/2	0.095	11.88	4.25	15090	23060	22170	29560	36950	46190	55430
	0.120	14.74	5.27	18730	28610	31210	41620	52020	65030	78030
1 5/8	0.049	6.87	2.46	8730	13340	14560	19410	24260	30330	36390
	0.058	8.09	2.89	10280	15700	17130	22840	28550	35690	42830

[1] The tensile strengths given in this table are the product of the tube areas by the unit stresses at the top of the columns. No reduction has been made for rivet or bolt holes or for the efficiency of a welded joint.

TABLE 9 : 2 (*Continued*)

WEIGHTS AND TENSILE STRENGTHS OF STANDARD ROUND TUBES[1]

Size		Weight per 100 Inch		Tensile Strengths, Pounds per Square Inch						
Diameter, Inches	t	Steel	Aluminum Alloy	36000	55000	60000	80000	100000	125000	150000
1 3/4	0.049	7.42 lb.	2.65 lb.	9420	14400	15710	20940	26180	32730	39270
	0.058	8.73	3.12	11100	16960	18500	24660	30830	38540	46250
	0.065	9.75	3.48	12390	18930	20650	27530	34410	43010	51620
	0.083	12.32	4.40	15650	23910	26080	34780	43470	54340	65210
	0.095	13.99	5.00	17780	27160	29630	39510	49390	61740	74090
	0.120	17.41	6.22	22120	33800	36870	49160	61450	76810	92180
1 7/8	0.058	9.38	3.35	11920	18210	19870	26490	33110	41390	49670
	0.065	10.47	3.74	13310	20330	22180	29570	36960	46200	55440
2	0.058	10.03	3.58	12740	19460	21230	28310	35390	44240	53090
	0.065	11.19	4.00	14220	21730	23710	31610	39510	49390	59270
	0.083	14.16	5.06	18000	27490	29990	39990	49990	62490	74990
	0.095	16.11	5.76	20470	31270	34110	45480	56850	71060	85280
2 1/4	0.065	12.64	4.52	16060	24540	26770	35700	44620	55780	66930
	0.083	16.01	5.72	20340	31080	33910	45210	56510	70640	84770
	0.095	18.22	6.51	23160	35380	38590	51460	64320	80400	96480
	0.120	22.75	8.13	28910	44170	48180	64240	80300	100370	120450
2 1/2	0.083	17.85	6.38	22690	34660	37810	50420	63020	78780	94530
	0.095	20.34	7.27	25840	39480	43070	57420	71780	89730	107670
2 3/4	0.083	19.70	7.04	25030	38250	41720	55630	69540	86930	104310
	0.095	22.48	8.03	28560	43640	47600	63470	79340	99180	119010
	0.120	28.09	10.04	35690	54530	59490	79320	99150	123940	148730
3	0.095	24.56	8.78	31210	47690	52020	69360	86700	108380	130050
	0.120	30.76	11.00	39090	59710	65140	86860	108570	135710	162860
3 1/4	0.120	33.43	11.95	42480	64900	70800	94400	118000	147500	177000
	5/32	43.02	15.38	54670	83520	91120	121490	151860	189830	227790
	3/16	51.11	18.27	64940	99220	108240	144320	180400	225500	270600
	1/4	66.75	23.86	84820	129590	141370	188500	235620	294530	353430
3 3/4	3/16	59.45	21.25	75550	115420	125910	167880	209850	262310	314780
	1/4	77.88	27.84	98960	151190	164930	219910	274890	343610	412340

[1] The tensile strengths given in this table are the product of the tube areas by the unit stresses at the top of the columns. No reduction has been made for rivet or bolt holes or for the efficiency of a welded joint.

TABLE 9:3
Areas of Tubes

Outside Diameter, Inches — B. W. G.	22	20	18	17	16	14	13	11			
t	0.028	0.035	0.049	0.058	0.065	0.083	0.095	0.120	5/32	3/16	1/4
1/4	0.01953	0.02364	0.03094	0.03499	0.03778	0.04355	0.04626	0.04901	....	....	....
3/8	0.03053	0.03739	0.05018	0.05776	0.06330	0.07614	0.08357	0.09613	0.1074	0.1104	....
1/2	0.04152	0.05113	0.06943	0.08054	0.08883	0.10873	0.12087	0.14326	0.1687	0.1841	0.1964
5/8	0.05252	0.06487	0.08867	0.10331	0.11435	0.14133	0.15818	0.19038	0.2301	0.2577	0.2945
3/4	0.06351	0.07862	0.10791	0.12609	0.13988	0.17392	0.19549	0.2375	0.2915	0.3313	0.3927
7/8	0.07451	0.09236	0.12715	0.14887	0.16541	0.2065	0.2328	0.2846	0.3528	0.4050	0.4909
1	0.08550	0.10611	0.14640	0.17164	0.19093	0.2391	0.2701	0.3318	0.4142	0.4786	0.5890
1 1/8	0.09650	0.11985	0.16564	0.19442	0.2165	0.2717	0.3074	0.3789	0.4755	0.5522	0.6872
1 1/4	0.10749	0.13360	0.18488	0.2172	0.2420	0.3043	0.3447	0.4260	0.5369	0.6259	0.7854
1 3/8	0.11849	0.14734	0.2041	0.2400	0.2675	0.3369	0.3820	0.4731	0.5983	0.6995	0.8836
1 1/2	0.12948	0.16109	0.2234	0.2628	0.2930	0.3695	0.4193	0.5202	0.6596	0.7731	0.9817
1 5/8	0.14048	0.17483	0.2426	0.2855	0.3186	0.4021	0.4566	0.5674	0.7210	0.8468	1.0799
1 3/4	0.15148	0.18857	0.2618	0.3083	0.3441	0.4347	0.4939	0.6145	0.7823	0.9204	1.1781
1 7/8	0.16247	0.2023	0.2811	0.3311	0.3696	0.4673	0.5312	0.6616	0.8437	0.9940	1.2763
2	0.17347	0.2161	0.3003	0.3539	0.3951	0.4999	0.5685	0.7087	0.9050	1.0677	1.3744
2 1/4	0.19546	0.2436	0.3388	0.3994	0.4462	0.5651	0.6432	0.8030	1.0278	1.2149	1.5708
2 1/2	0.2174	0.2710	0.3773	0.4450	0.4972	0.6302	0.7178	0.8972	1.1505	1.3622	1.7671
2 3/4	0.2394	0.2985	0.4158	0.4905	0.5483	0.6954	0.7934	0.9915	1.2732	1.5094	1.9635
3	0.2614	0.3260	0.4543	0.5361	0.5993	0.7606	0.8670	1.0857	1.3959	1.6567	2.1598
3 1/4	0.2834	0.3535	0.4928	0.5816	0.6504	0.8258	0.9416	1.1800	1.5186	1.8040	2.3562
3 1/2	0.3054	0.3810	0.5312	0.6272	0.7014	0.8910	1.0162	1.2742	1.6414	1.9512	2.5525
3 3/4	0.3274	0.4085	0.5697	0.6727	0.7525	0.9562	1.0908	1.3685	1.7641	2.0985	2.7489
4	0.3494	0.4360	0.6082	0.7183	0.8035	1.0214	1.1655	1.4627	1.8868	2.2457	2.9452

TABLE 9 : 4

I of Hollow Circles and Tubes

Outside Diameter, Inches — B. W. G.	22	20	18	17	16	14	13	11			
t	0.028	0.035	0.049	0.058	0.065	0.083	0.095	0.120	5/32	3/16	1/4
1/4	0.0001222	0.0001402	0.0001655	0.0001759	0.0001816	0.0001893	0.0001911	0.0001917	....	....	....
3/8	0.0004600	0.0005459	0.0006817	0.0007498	0.0007939	0.0008771	0.0009132	0.0009544	0.0009700	0.0009707	....
1/2	0.0011603	0.0013898	0.0017860	0.002001	0.002148	0.002457	0.002615	0.002844	0.003007	0.003056	0.003068
5/8	0.002345	0.002833	0.003704	0.004195	0.004543	0.005311	0.005733	0.006412	0.007022	0.007298	0.007478
3/4	0.004145	0.005036	0.006661	0.007601	0.008278	0.009822	0.010704	0.012211	0.013733	0.014561	0.015340
7/8	0.006689	0.008161	0.010882	0.012484	0.013653	0.016370	0.017966	0.02079	0.02386	0.02571	0.02780
1	0.010106	0.012368	0.016594	0.019111	0.02097	0.02534	0.02796	0.03271	0.03812	0.04160	0.04602
1 1/8	0.014525	0.017818	0.02402	0.02775	0.03052	0.03711	0.04111	0.04852	0.05724	0.06310	0.07114
1 1/4	0.02008	0.02467	0.03339	0.03867	0.04260	0.05206	0.05787	0.06876	0.08192	0.09107	0.10431
1 3/8	0.02688	0.03309	0.04492	0.05213	0.05753	0.07059	0.07867	0.09400	0.11290	0.12637	0.14669
1 1/2	0.03508	0.04324	0.05885	0.06841	0.07558	0.09305	0.10394	0.12478	0.15089	0.16988	0.19942
1 5/8	0.04480	0.05528	0.07540	0.08776	0.09707	0.11985	0.13413	0.16166	0.19661	0.2224	0.2637
1 3/4	0.05616	0.06936	0.09478	0.11046	0.12230	0.15136	0.16967	0.2052	0.2508	0.2849	0.3405
1 7/8	0.06928	0.08565	0.11944	0.13677	0.15156	0.18797	0.2110	0.2559	0.3141	0.3582	0.4312
2	0.08434	0.10432	0.14299	0.16696	0.18514	0.2300	0.2586	0.3141	0.3873	0.4431	0.5369
2 1/4	0.12065	0.14940	0.2052	0.2401	0.2665	0.3322	0.3741	0.4568	0.5663	0.6514	0.7977
2 1/2	0.16612	0.2059	0.2834	0.3318	0.3688	0.4607	0.5197	0.6369	0.7934	0.9165	1.1320
2 3/4	0.2218	0.2751	0.3793	0.4446	0.4944	0.6189	0.6991	0.8590	1.0746	1.2456	1.5493
3	0.2887	0.3583	0.4946	0.5802	0.6457	0.8097	0.9156	1.1276	1.4154	1.6454	2.0587
3 1/4	0.3678	0.4568	0.6313	0.7410	0.8251	1.0361	1.1727	1.4472	1.8216	2.1228	2.6691
3 1/2	0.4602	0.5718	0.7910	0.9291	1.0349	1.3012	1.4739	1.8220	2.2989	2.6848	3.3901
3 3/4	0.5670	0.7048	0.9756	1.1465	1.2777	1.6080	1.8228	2.2565	2.8533	3.3383	4.2307
4	0.6891	0.8568	1.1870	1.3955	1.5557	1.9597	2.2228	2.7552	3.4903	4.0902	5.2002

The polar moment of inertia is equal to twice the value given in the above table.

The radius of gyration $\rho = \sqrt{\frac{I}{A}}$. For round tubes ρ = approximately $D/3$ or, more closely, $0.35\ D$, where D is the outside diameter.

TABLE 9 : 5

VALUES OF I/y OF TUBES

Outside Diameter, Inches — B. W. G.	22	20	18	17	16	14	13	11			
t	0.028	0.035	0.049	0.058	0.065	0.083	0.095	0.120	5/32	3/16	1/4
1/4	0.000978	0.001122	0.001324	0.001407	0.001453	0.001514	0.001529	0.001534	....	....	....
3/8	0.002463	0.002912	0.003636	0.003999	0.004234	0.004678	0.004871	0.005090	0.00517	0.00518	....
1/2	0.004641	0.005559	0.007144	0.008003	0.008592	0.009828	0.010459	0.011375	0.01203	0.01222	0.01227
5/8	0.007503	0.009065	0.011852	0.013425	0.014538	0.016996	0.018344	0.02052	0.02247	0.02336	0.02393
3/4	0.011052	0.013429	0.017762	0.02027	0.02208	0.02619	0.02854	0.03256	0.03662	0.03883	0.04091
7/8	0.015289	0.018653	0.02487	0.02853	0.03121	0.03742	0.04107	0.04753	0.05454	0.05876	0.06355
1	0.02021	0.02474	0.03319	0.03822	0.04193	0.05068	0.05591	0.06542	0.07624	0.08320	0.09204
1 1/8	0.02582	0.03168	0.04270	0.04933	0.05425	0.06597	0.07309	0.08625	0.10175	0.11217	0.12647
1 1/4	0.03212	0.03948	0.05342	0.06187	0.06816	0.08330	0.09259	0.11002	0.13108	0.14571	0.16690
1 3/8	0.03911	0.04814	0.06534	0.07583	0.08367	0.10267	0.11443	0.13673	0.16422	0.18382	0.2134
1 1/2	0.04678	0.05765	0.07847	0.09121	0.10079	0.12407	0.13859	0.16638	0.2012	0.2265	0.2659
1 5/8	0.05514	0.06803	0.09279	0.10801	0.11948	0.14751	0.16508	0.19897	0.2420	0.2738	0.3245
1 3/4	0.06418	0.07927	0.10832	0.12624	0.13977	0.17299	0.19391	0.2345	0.2866	0.3256	0.3892
1 7/8	0.07390	0.09136	0.12740	0.14589	0.16166	0.2005	0.2251	0.2727	0.3351	0.2821	0.4600
2	0.08434	0.10432	0.14299	0.16696	0.18514	0.2301	0.2586	0.3144	0.3873	0.4431	0.5369
2 1/4	0.10724	0.13280	0.18246	0.2134	0.2369	0.2953	0.3325	0.4060	0.5034	0.5790	0.7090
2 1/2	0.13290	0.16469	0.2267	0.2655	0.2950	0.3686	0.4158	0.5095	0.6348	0.7322	0.9056
2 3/4	0.16130	0.2001	0.2759	0.3233	0.3596	0.4501	0.5084	0.6248	0.7815	0.9059	1.1268
3	0.19245	0.2389	0.3298	0.3868	0.4305	0.5398	0.6104	0.7518	0.9436	1.0969	1.3724
3 1/4	0.2264	0.2811	0.3885	0.4560	0.5077	0.6376	0.7217	0.8906	1.1210	1.3064	1.6425
3 1/2	0.2630	0.3268	0.4520	0.5309	0.5914	0.7435	0.8422	1.0411	1.3137	1.5342	1.9372
3 3/4	0.3024	0.3759	0.5203	0.6115	0.6814	0.8576	0.9722	1.2035	1.5217	1.7804	2.2564
4	0.3445	0.4284	0.5935	0.6978	0.7779	0.9799	1.1114	1.3776	1.7452	2.0451	2.6001

The polar section modulus is equal to twice the value given in the above table.

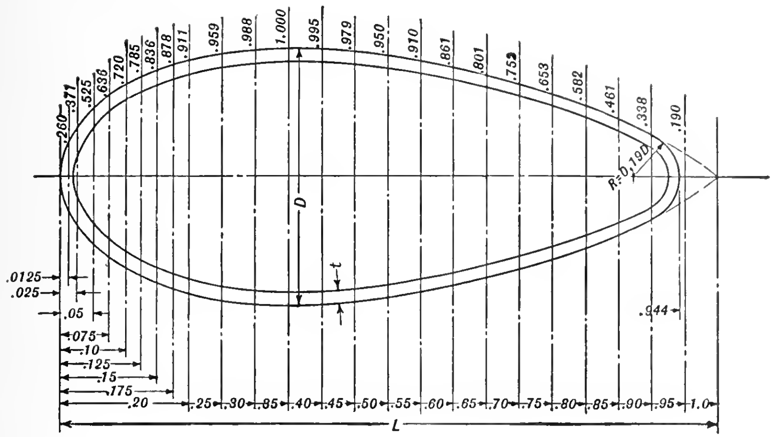


TABLE 9 : 6

PROPERTIES OF SEAMLESS STREAMLINE TUBES

Basic Round Tube			Drawn Streamline Tube			
O. D.	Gage	Area	Major Axis	Minor Axis	I_{xx}	ρ_{xx}
1 1/8	0.035	0.1199	1.517	0.643	0.00608	0.2251
1 1/4[1]	0.035	0.1336	1.685	0.714	0.00841	0.2509
	0.049	0.1849			0.01139	0.2482
1 3/8[1]	0.035	0.1473	1.855	0.786	0.01128	0.2767
	0.049	0.2041			0.01532	0.2740
1 1/2[1]	0.049	0.2234	2.023	0.857	0.02007	0.2997
	0.058	0.2628			0.02333	0.2979
1 5/8	0.049	0.2426	2.192	0.929	0.02571	0.3255
	0.058	0.2855			0.02993	0.3238
1 3/4[1]	0.049	0.2618	2.360	1.000	0.03232	0.3514
	0.058	0.3083			0.03767	0.3496
	0.065	0.3441			0.04170	0.3481
1 7/8	0.049	0.2811	2.528	1.071	0.04073	0.3807
	0.058	0.3311			0.04664	0.3754
	0.065	0.3696			0.05168	0.3739
2[1]	0.058	0.3539	2.697	1.143	0.05693	0.4011
	0.065	0.3951			0.06313	0.3997
2 1/4	0.058	0.3994	3.035	1.286	0.08187	0.4527
	0.065	0.4462			0.09088	0.4513
2 1/2	0.065	0.4972	3.372	1.429	0.12576	0.5029
	0.083	0.6302			0.15710	0.4993
2 3/4	0.065	0.5483	3.708	1.571	0.16859	0.5545
	0.083	0.6954			0.21104	0.5509
3[1]	0.065	0.5993	4.045	1.714	0.22018	0.6061
	0.083	0.7606			0.27611	0.6025

[1] On the 1926 Tentative Standard List.

When a tension member is pierced by rivet or bolt holes, that portion of the cross-section occupied by the holes must be subtracted from the gross sectional area to obtain the net area. When making this computation it is desirable to assume the holes 1/32 to 1/16 in. larger than their nominal size to allow for injury to the metal in forming the holes and also for local concentration of stress.

9 : 2. Types of Failure in Compression Members — It has been shown in the study of Mechanics of Materials that an approximately cubical prism subjected to a compressive stress uniformly distributed over its cross-section will fail when the intensity of stress reaches a certain critical value. If the material is ductile it will flow and thus change its position, whereas if it is brittle it may shear and rupture. If the load is removed the material will not make more than a partial return to its original position even though it is highly elastic, and the member will be permanently deformed. This type may be called "plastic" failure since it involves plastic deformation in the member. The critical unit stress at which such failure occurs is, usually, the compression yield point of the material.

A compression member having a high ratio of length to width or thickness, however, will fail under a unit stress much less than the yield point. When the load reaches a certain magnitude the whole member will bend laterally and behave very much like a spring. If the load is reduced only slightly below this magnitude, the member will return to its original position and a careful examination will show it to be uninjured. This type of failure depends on the modulus of elasticity of the material rather than on its compressive strength and it is therefore called "elastic" failure or "buckling."

When a long column is loaded so that it fails elastically and bends laterally, the unit stresses on the concave side of the axis of the member are increased by compression due to bending, and on the convex side they are decreased. If the deflection is permitted to continue, the maximum intensity of compressive stress will pass the yield point of the material and the elastic failure will be accompanied by a plastic failure. Once plastic failure has started, removal of the load will not suffice for the return of the member to its original position.

The kinds of failure just described are typical for very short and for very long columns, but a large majority of the columns used in conventional structures are of intermediate length. Their failures, therefore, show a combination of the characteristics of those of the extreme types. Their behavior under practical conditions, however, is so variable that large differences in strength are often found between two apparently identical columns tested under apparently identical con-

ditions. As a consequence there are nearly as many formulas for predicting the strengths of columns of normal length as there have been investigators of compression members.

9 : 3. Euler's Formula for Long Columns — For columns that fail elastically, the formula developed by the Swiss mathematician, Euler, is preëminent. It may be written in terms of the critical load:

$$P = \frac{c\pi^2 EI}{L^2} \qquad 9:1$$

or in terms of the critical average unit stress:

$$\frac{P}{A} = \frac{c\pi^2 E}{(L/\rho)^2} \qquad 9:2$$

where P = the critical load in pounds.

c = a coefficient dependent on the method of fastening the ends of the column.

E = the modulus of elasticity of the material in pounds per square inch.

I = the least moment of inertia of the cross-section about an axis through its centroid, in inches.[4]

L = the column length in inches.

A = the cross-sectional area in square inches.

$\rho = \sqrt{\frac{I}{A}}$ = the radius of gyration of the cross-section in inches.

π = 3.1416.

The ratio, L/ρ is generally called the slenderness ratio of a column. The derivation of Euler's formula is given in practically all texts on Mechanics of Materials.[1] Tests have shown that when the slenderness ratio is so large that the critical unit stress, P/A, indicated by it is less than about half the yield point of the material, this unit stress is very close to that at which failure actually occurs. Therefore Euler's formula is generally used in practical airplane design for columns of this class.

9 : 4. Formulas for "Short" Columns — Columns of the dimensions ordinarily encountered in structural design are generally spoken of as "short" columns, though it would be more correct to term them columns of intermediate length. No single formula fits all of the test data available on short columns of various shapes and sizes and of

[1] Fuller and Johnston, "Applied Mechanics," Vol. II, pages 346–354. Seeley, "Resistance of Materials," Art. 95. Swain, "Strength of Materials," page 420. Poorman, "Strength of Materials," Chap. XIV. Morley, "Strength of Materials," Arts. 100–101.

various materials. Experience has indicated, however, that for most aeronautical work satisfactory results can be obtained from J. B. Johnson's Parabolic formula:

$$\frac{P}{A} = F - \frac{F^2(L/\rho)^2}{4\,c\pi^2 E} \qquad 9:3$$

where F is the yield point of the material and the other symbols have the same significance as in the Euler formula.

If Formula 9 : 3 is plotted with values of P/A as ordinates and L/ρ as abscissas, the curve will be a parabola with its vertex at $P/A = F$, $L/\rho = 0$, and will be tangent to the curve of the Euler formula at the value of L/ρ where $P/A = F/2$. This value of L/ρ is sometimes called the "critical" slenderness ratio as it marks the practical dividing line between "short" and "long" or "Euler" columns. It is not a fixed value but depends on the values of F, E and c.

The combination of the curve of the parabolic formula for slenderness ratios smaller, and the curve of the Euler formula for slenderness ratios larger, than the critical is known as the "Parabolic-Euler" column curve. It is used in airplane design for all types of columns except aluminum alloy tubes.

In the design of aluminum alloy tubes as compression members, the "Straight-line Euler" column curve is used. In this combination the parabola is replaced by a straight line the equation of which is

$$\frac{P}{A} = 48{,}000 - \frac{400\,L}{\rho\sqrt{c}} \qquad 9:4$$

This equation represents a line passing through $P/A = 48{,}000$, $L/\rho = 0$, and tangent to the Euler curve for aluminum alloy and the same value of c. It is specified that when this formula indicates that the allowable value of P/A is greater than 40,000 lb. per sq. in., 40,000 should be used.

Table 9 : 7 gives the column formulas used with the more common materials employed in airplane design, the proper constants having been substituted in the general Euler, Parabolic, and Straight-line formulas. In each case the formulas for both the total load P and the unit stress P/A have been given. Several sets of formulas are given for steel as the yield points vary with the heat-treatment and chemical content, thus changing the constants in the parabolic formula and the values of the critical slenderness ratios, but making little change in the Euler formulas.

The differences shown in the Euler formulas are due to the fact that for steels with low yield points the modulus of elasticity is usually only 28,000,000 lb. per sq. in., whereas for the grades of steel that can

be heat-treated to yield points of 60,000 lb. per sq. in., or better, the modulus of elasticity will be 29,000,000 lb. per sq. in. For each material and yield point represented in Table 9 : 7, formulas are given for two values of the restraint coefficient, c, a variable that is discussed in the next article.

TABLE 9 : 7

COLUMN FORMULAS FOR USE IN DESIGN

Material	c	Short Columns	Critical L/ρ	Long Columns
Spruce	1	$P/A = 5{,}000 - 0.500(L/\rho)^2$	72	$P/A = 13{,}000{,}000/(L/\rho)^2$
	1	$P = 5{,}000A - 0.500\, L^2A^2/I$		$P = 13{,}000{,}000\, I/L^2$
	2	$P/A = 5{,}000 - 0.250\,(L/\rho)^2$	102	$P/A = 26{,}000{,}000/(L/\rho)^2$
	2	$P = 5{,}000A - 0.250\, L^2A^2/I$		$P = 26{,}000{,}000\, I/L^2$
Aluminum alloy[1]	1	$P/A = 48{,}000 - 400\, L/\rho$	80	$P/A = 104{,}000{,}000/(L/\rho)^2$
	1	$P = 48{,}000A - 400\, LA/\rho$		$P = 104{,}000{,}000\, I/L^2$
	2	$P/A = 48{,}000 - 280\, L/\rho$	114	$P/A = 208{,}000{,}000/(L/\rho)^2$
	2	$P = 48{,}000A - 280\, LA/\rho$		$P = 208{,}000{,}000\, I/L^2$
Steel Commercial Y.P. 27,000	1	$P/A = 27{,}000 - 0.660\,(L/\rho)^2$	143	$P/A = 276{,}000{,}000/(L/\rho)^2$
	1	$P = 27{,}000A - 0.660\, L^2A^2/I$		$P = 276{,}000{,}000\, I/L^2$
	2	$P/A = 27{,}000 - 0.330\,(L/\rho)^2$	202	$P/A = 552{,}000{,}000/(L/\rho)^2$
	2	$P = 27{,}000A - 0.330\, L^2A^2/I$		$P = 552{,}000{,}000\, I/L^2$
Y.P. 36,000	1	$P/A = 36{,}000 - 1.172\,(L/\rho)^2$	124	$P/A = 276{,}000{,}000/(L/\rho)^2$
	1	$P = 36{,}000A - 1.172\, L^2A^2/I$		$P = 276{,}000{,}000\, I/L^2$
	2	$P/A = 36{,}000 - 0.586\,(L/\rho)^2$	175	$P/A = 552{,}000{,}000/(L/\rho)^2$
	2	$P = 36{,}000A - 0.586\, L^2A^2/I$		$P = 552{,}000{,}000\, I/L^2$
Y.P. 60,000	1	$P/A = 60{,}000 - 3.144\,(L/\rho)^2$	98	$P/A = 286{,}000{,}000/(L/\rho)^2$
	1	$P = 60{,}000A - 3.144\, L^2A^2/I$		$P = 286{,}000{,}000\, I/L^2$
	2	$P/A = 60{,}000 - 1.572\,(L/\rho)^2$	138	$P/A = 572{,}000{,}000/(L/\rho)^2$
	2	$P = 60{,}000A - 1.572\, L^2A^2/I$		$P = 572{,}000{,}000\, I/L^2$
Y.P. 105,000	1	$P/A = 105{,}000 - 9.620\,(L/\rho)^2$	74	$P/A = 286{,}000{,}000/(L/\rho)^2$
	1	$P = 105{,}000A - 9.620\, L^2A^2/I$		$P = 286{,}000{,}000\, I/L^2$
	2	$P/A = 105{,}000 - 4.810\,(L/\rho)^2$	105	$P/A = 572{,}000{,}000/(L/\rho)^2$
	2	$P = 105{,}000A - 4.810\, L^2A^2/I$		$P = 572{,}000{,}000\, I/L^2$

[1] Where these formulas indicate an allowable P/A greater than 40,000 lb. per sq. in., 40,000 shall be used. Table from "Handbook of Instructions for Airplane Designers," U. S. Army Air Corps, Modified.

The formulas in this table apply only to columns on which the load is applied without any intentional eccentricities. The case of a column with intentional eccentricity of loading is discussed in Chapter XI on "Combined Bending and Axial Load."

None of the other numerous column formulas that are in common use in civil engineering are discussed here, either because they are not applicable to airplane design, or they do not have the simplicity or close agreement with test data that are characteristic of the Parabolic-Euler and Straight-line Euler combinations.

9 : 5. End Conditions — A factor of the utmost importance in determining the load which a column can carry is the restraint imposed by

the structure to which it is connected upon any tendency of the ends of the column to move laterally or to rotate. If the load on a column were transmitted to it through fixed frictionless pins or knife-edges with their axes intersecting that of the column at right angles, the ends of the latter would be free to rotate though fixed in position. Some writers describe such a column as one with "position fixed ends" but it is more often termed a "theoretically pin-ended column." When subjected to load it would bend in a simple curve similar to that shown in Fig. 9 : 1, and its "restraint coefficient," c, would be unity.

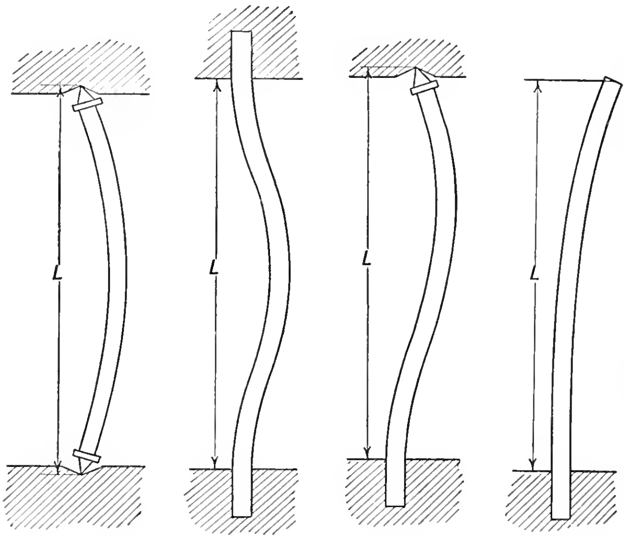


Fig. 9 : 1 Fig. 9 : 2 Fig. 9 : 3 Fig. 9 : 4

If the ends were connected rigidly to a structure so stiff that the axis of the column was fixed in direction as well as position at both ends, it would bend in a doubly reversed curve similar to that shown in Fig. 9 : 2. Such a column would be said to have "direction fixed ends" but it is more commonly known as a "theoretically fixed-ended column." Its restraint coefficient would be 4. Other theoretical end conditions are "one end pinned and one end fixed," shown in Fig. 9 : 3, for which the restraint coefficient is 2.25, and "one end fixed and one end free," shown in Fig. 9 : 4, for which the restraint coefficient is 0.25.

All of these conditions are purely theoretical and apply strictly to ideal columns only. Frictionless pins do not exist, and column ends cannot be absolutely fixed in direction as they will always rotate slightly even though they are firmly connected to a very heavy structure. For all columns met with in airplane design the true end conditions fall

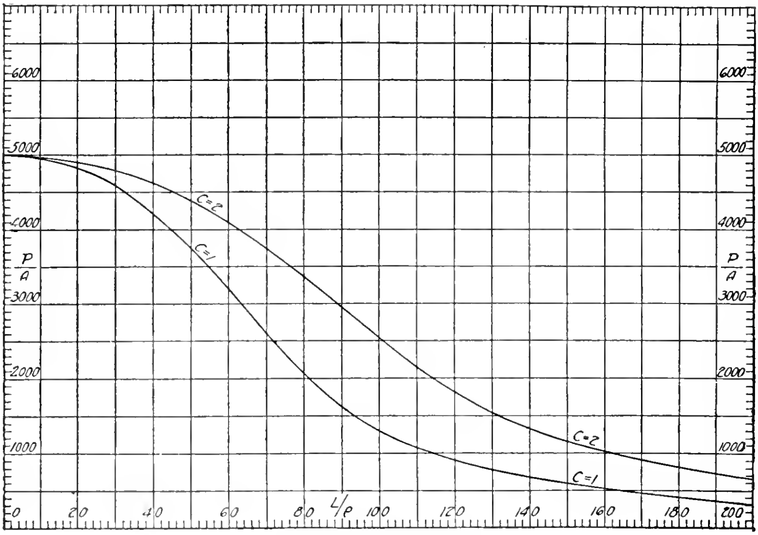
somewhere between the theoretical pin-ended condition, for which $c = 1$, and the theoretical fixed-ended condition, for which $c = 4$. Where the end connection is made with a physical pin the restraint is so little more than with the theoretical pin-ended column that it is customary to use $c = 1$ in design. When a column is rigidly connected to the rest of the structure, as for example a tube in a welded steel fuselage, the restraint conditions are much better than those represented by $c = 1$, and a higher value may be safely used. In airplane construction the ends of a column are never held so rigidly that it would be allowable to assume theoretically fixed ends and use $c = 4$ in design. Experience based upon static tests of a number of structures has indicated that $c = 2$ is the maximum that can be relied upon. Even this value should be used only where the joints are very rigid, as in welded construction. For riveted joints it is better to use a value of c not larger than 1.5 unless tests prove that $c = 2$ may be safely used.

In the Euler formula the critical load varies in direct proportion to the restraint coefficient, c, so the importance of the correct estimation of this quantity for long columns is obvious. In the parabolic and straight-line formulas the restraint coefficient appears only in the second and negative term which varies directly with the slenderness ratio L/ρ. As a result, the importance of a correct estimate of c decreases directly with the length of the column, vanishing when that length becomes zero. That this is reasonable is shown by the fact that when the column is very short it has little chance to bend, and therefore the shape of its deflection curve can make little difference in the load causing failure.

The fixed-ended strut shown in Fig. 9 : 2 reverses curvature at two points, and it has already been demonstrated in Mechanics of Materials that these are the quarter-points, and thus the distance between them is one-half the total length of the column. It has also been shown that these points of reversed curvature are locations where the bending moment is zero and the resultant load on the column passes through the centroid of the column cross-section. If we should isolate this central half of the column as a free body, it should therefore act as a pin-ended column. Hence the critical load for a fixed-ended column of length L, and that for a pin-ended column of length $L/2$ should be the same. A glance at Euler's formula should show that this is the case. A similar relation exists between a pin-ended column and any other column so restrained at the ends that the proper value of c is between 1.0 and 4.0. There will be restraining moments at the ends and also two intermediate points of inflection. The distance between the latter may be called the "equivalent pin-ended length" of the

column, and it will be equal to the actual length divided by the square root of c. Thus if $c = 2$, the equivalent pin-ended length will be $0.7071\,L$.

9 : 6. Practical Design of Columns — As with beams, the method of designing columns is to select a trial design and compute its margin of safety under the required load, continuing the process until the most satisfactory size has been found. If there are many sizes from which to select, this becomes a tedious process, but when it is intended



Column curves for spruce

FIG. 9 : 5

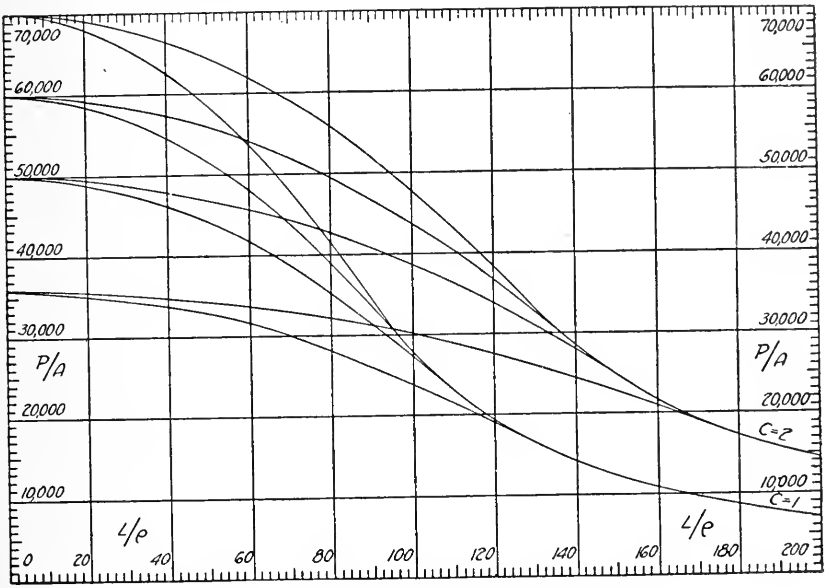
to use one of a limited number of standard sizes, tables or charts can be used that greatly reduce the labor.

When such tables or charts are not available, the first step after selecting a trial size is to compute its radius of gyration, ρ, and from that the slenderness ratio L/ρ. By comparing the latter value with the critical slenderness ratio for the material it can be seen whether the Euler or a short column formula should be used. The strength of the trial design can then be found by applying the proper formula for the case in hand. The critical slenderness ratios and column formulas for the materials most used in airplane design have been listed in Table 9 : 7.

This process can be simplified by the use of column curves with allowable unit stress, P/A, plotted against slenderness ratio, L/ρ. Once the slenderness ratio has been computed, the allowable unit stress

can be read off the column curve and multiplied by the sectional area of the column to determine its strength. Column curves for spruce, various qualities of steel, and aluminum alloy are given in Figs. 9 : 5 to 9 : 8.

At present the majority of columns used in airplane structures are round tubes of steel or aluminum alloy of which there are relatively



Column curves for steels of various yield points

E = 28,000,000 lb. per sq. in.

FIG. 9 : 6

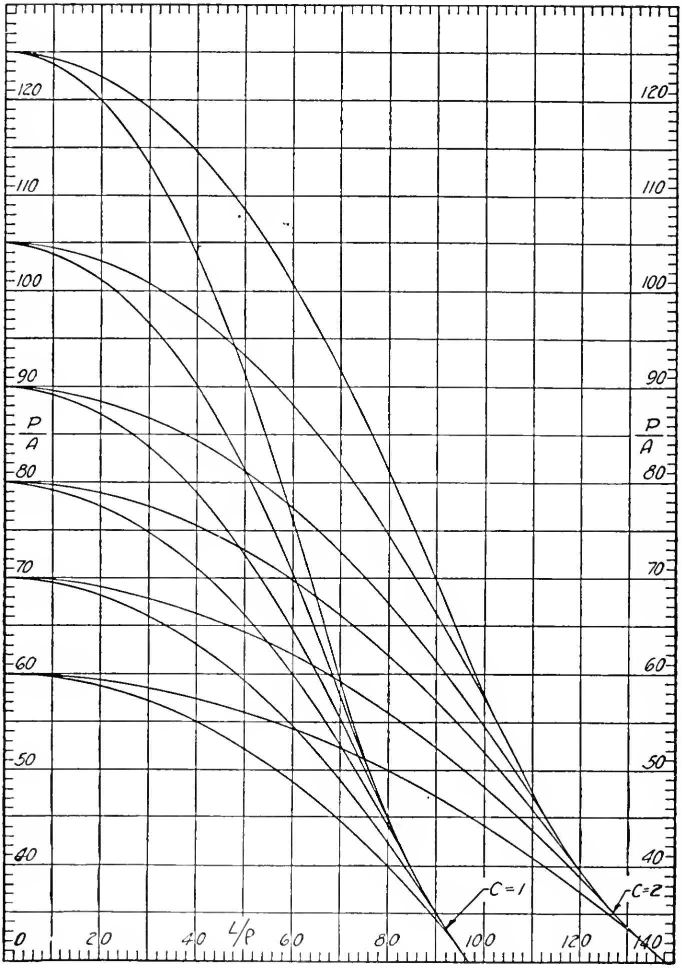
few sizes to be considered. To facilitate the work of the designer in selecting the tube to be used in any given location, the authors, while employed at McCook Field, developed the nomographic charts of Figs. 9 : 9 to 9 : 19. These charts are basically the various column formulas plotted according to the nomographic system instead of the usual rectangular coördinates.

Fig. 9 : 10 is a nomogram for Euler's formula for pin-ended steel columns.

$$P = \frac{\pi^2 EI}{L^2} \quad \text{where } c = 1.0$$

If a line is drawn from any point on the scale of lengths, L, to a point on the scale of loads, P, it will intersect the scale of moments of inertia, I, at the point representing the moment of inertia of the pin-ended

column of length, L, for which the load P is critical. Since there are two values of modulus of elasticity, E, in use for different grades of steel, the length scale is graduated on both sides, one set of graduations



Column curves for steels of various yield points

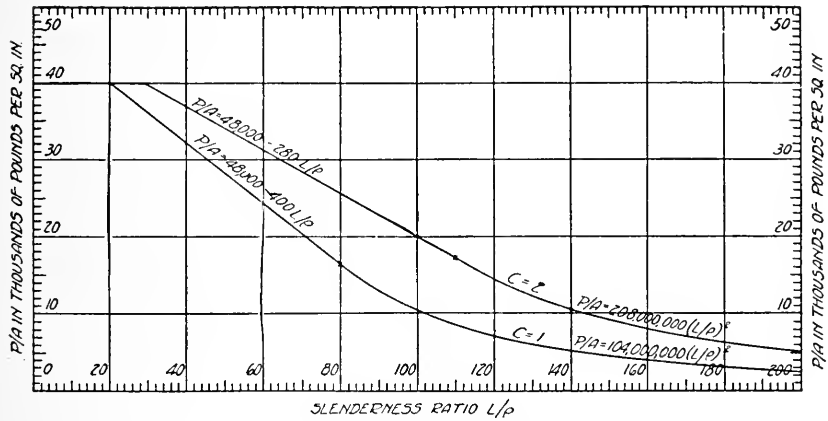
E = 29,000,000 lb. per sq. in.

FIG. 9 : 7

to be used for the plain carbon steels for which E is assumed as 28,000,000 lb. per sq. in., and the other for alloy steels for which E is assumed as 29,000,000 lb. per sq. in. The figure has two load and two moment of inertia scales. To get correct results the left-hand load scale must

be used in connection with the left-hand moment of inertia scale, the two right-hand scales also being used together. The object of this arrangement is purely to get as large a range of values as possible on the one chart.

While Fig. 9 : 10 was drawn specifically for pin-ended steel columns, it can be used for other materials or other end conditions with the aid of simple multipliers. If it is being used for some other material the load scales would be considered as modified by the ratio of E of the given material to that of steel. Thus for aluminum alloy approximate results can be obtained by considering the points on the load scales



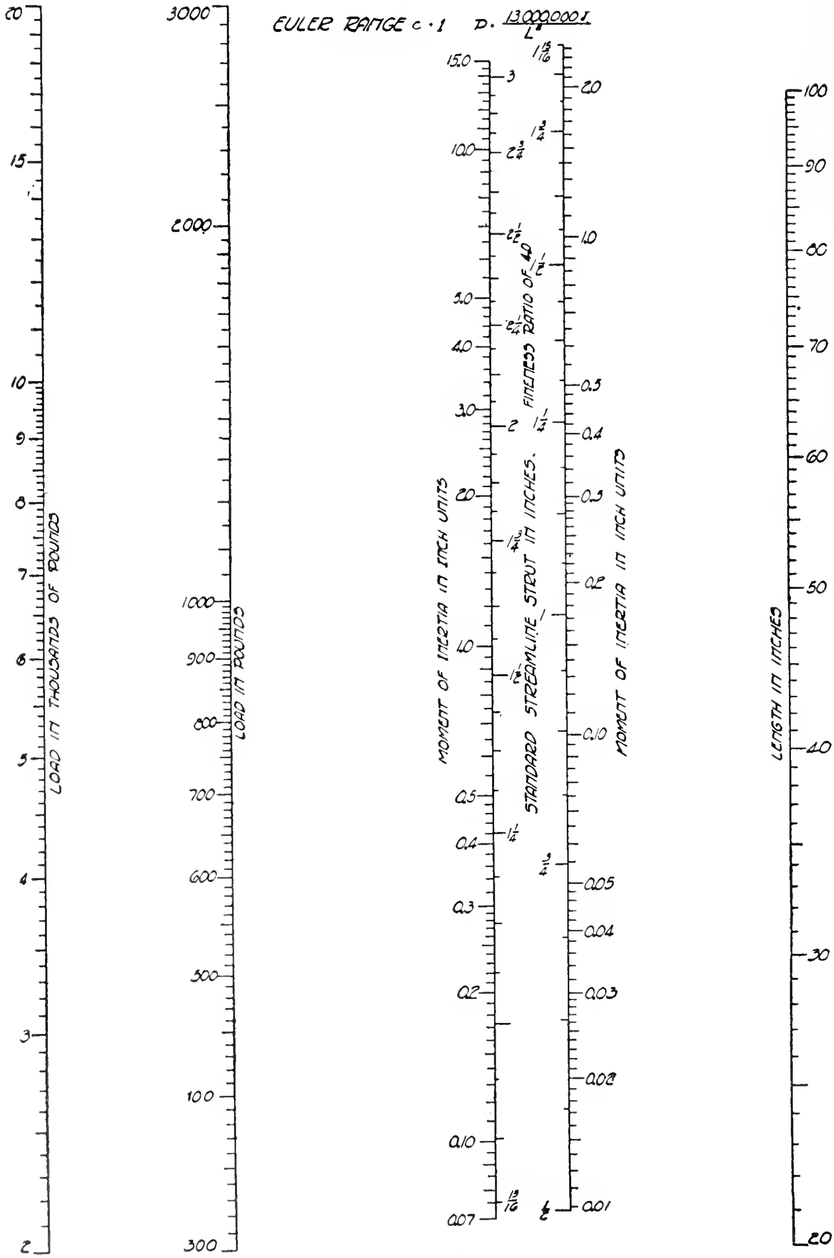
Column curves for aluminum alloy

Fig. 9 : 8

to represent only one-third their indicated values. For more precise work the length scale for $E = 28{,}000{,}000$ should be used, and the quantities indicated on the load scales divided by $28.0/10.4 = 2.7$ or multiplied by 0.372. The method of using it for values of c other than 1.0 is shown by an illustrative problem on page 152. For spruce members the chart of Fig. 9 : 9 would be used instead of employing multipliers.

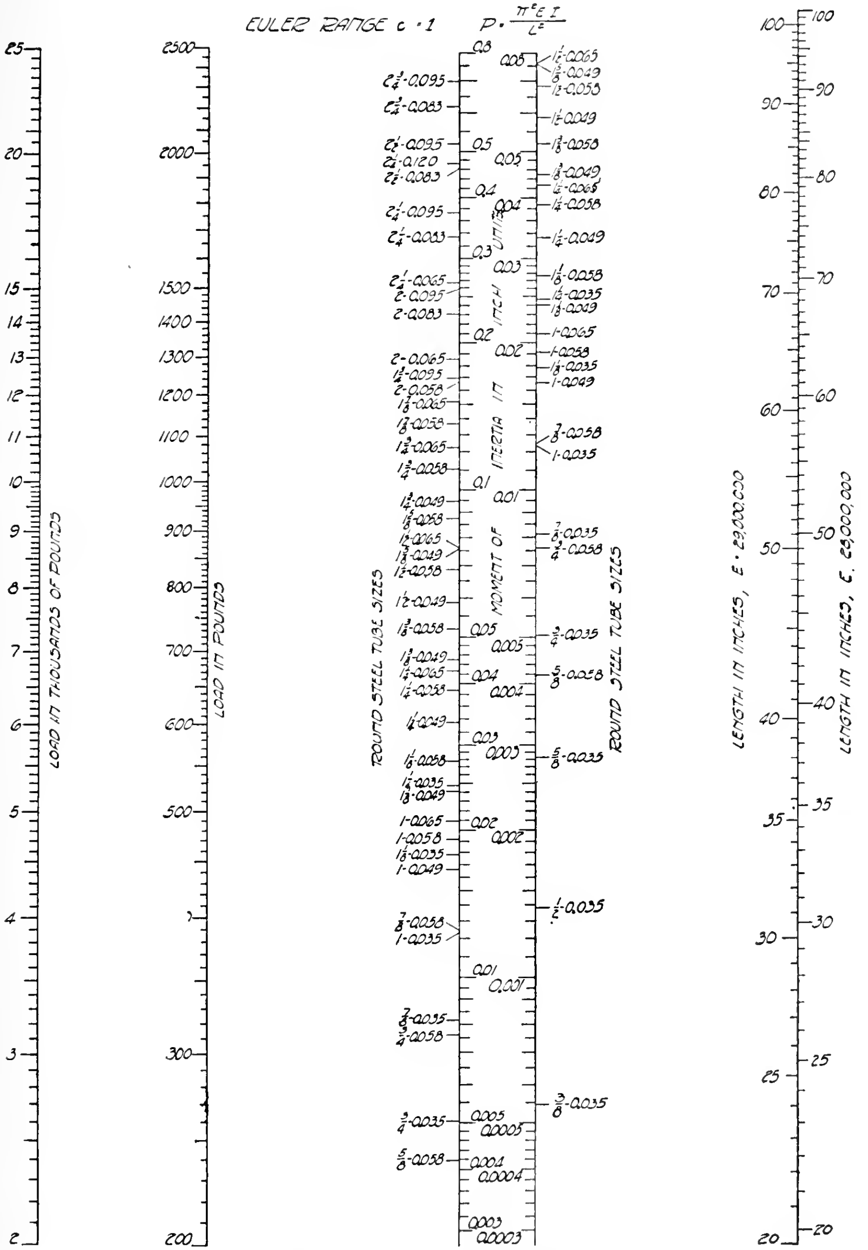
Figure 9 : 10 can be used for any steel or aluminum alloy member to which the Euler formula applies. As the type of member most used is the round tube, the moment of inertia scales are also graduated in tube sizes for the convenience of the designer. Similarly the I scale of Fig. 9 : 9 has been graduated for sizes of streamline spruce struts conforming to the Army Air Corps standard section and having a fineness ratio of 4.0.

Figures 9 : 11 to 9 : 18 are nomograms for steel tubes of lengths to which



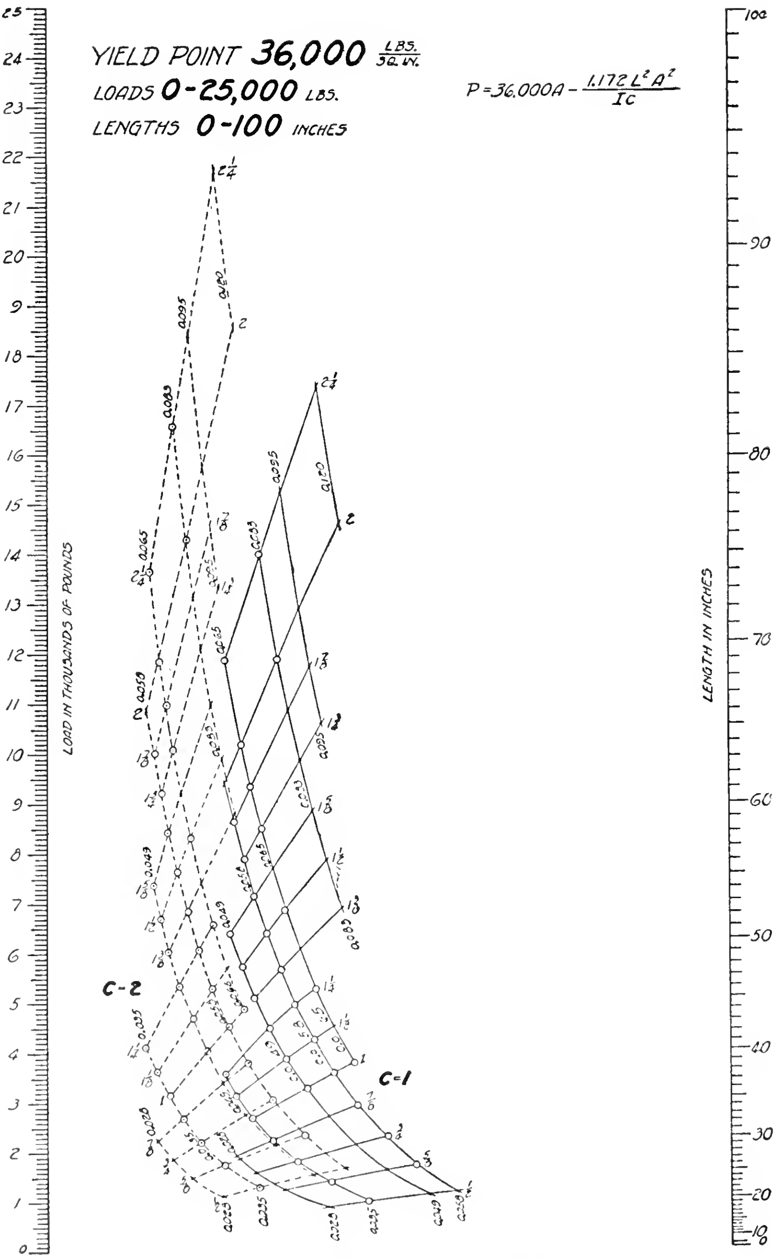
Nomogram for Spruce Struts, Euler Range

Fig. 9 : 9



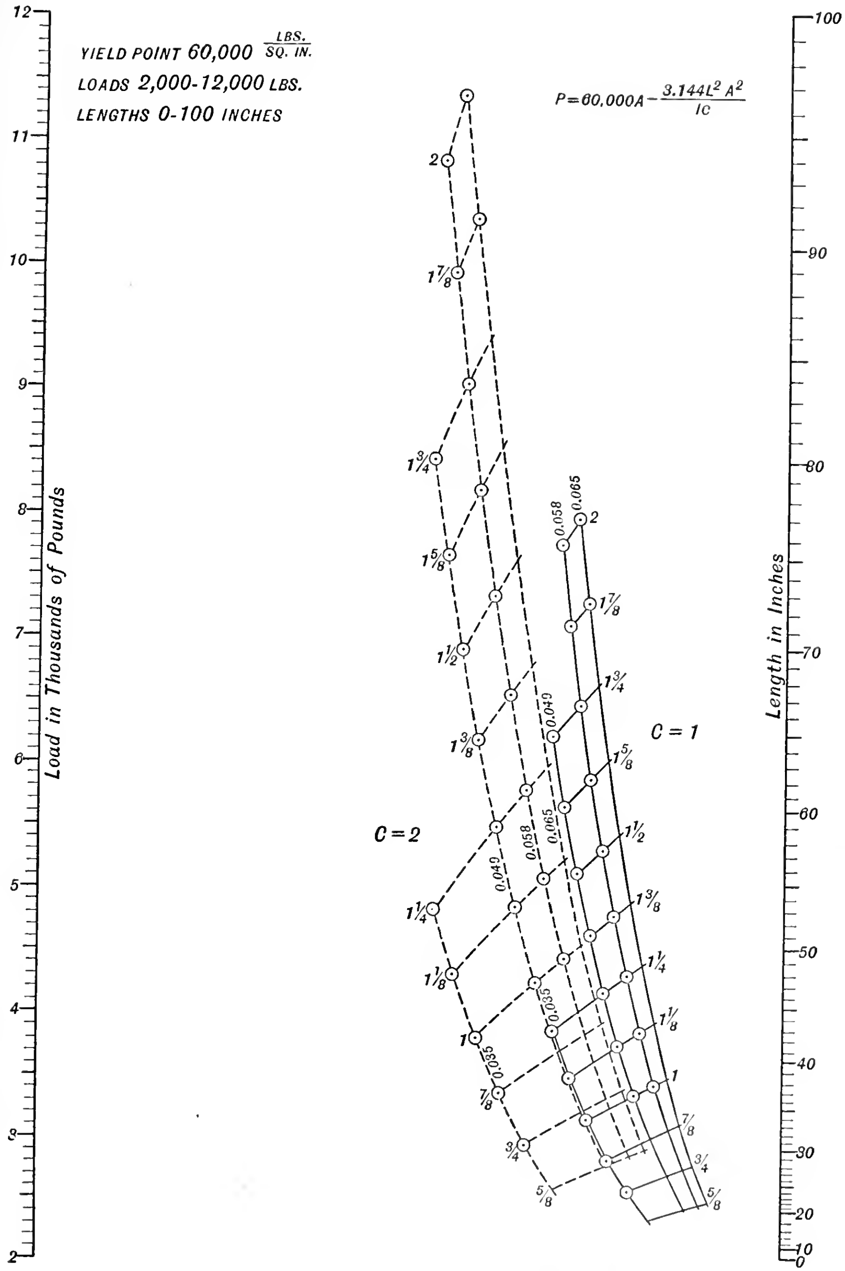
Nomogram for Steel Tubes, Euler Range

FIG. 9 : 10



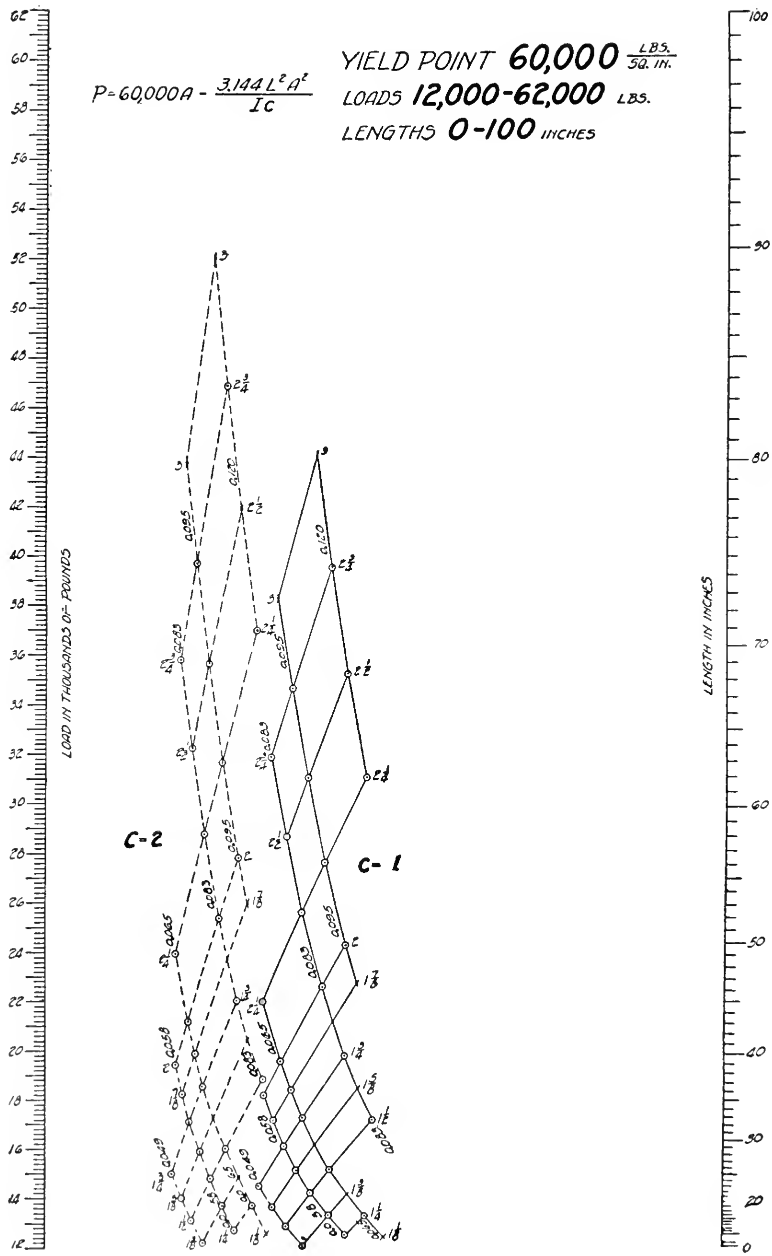
Nomogram for Steel Tubes, Short Column Range

FIG. 9 : 11



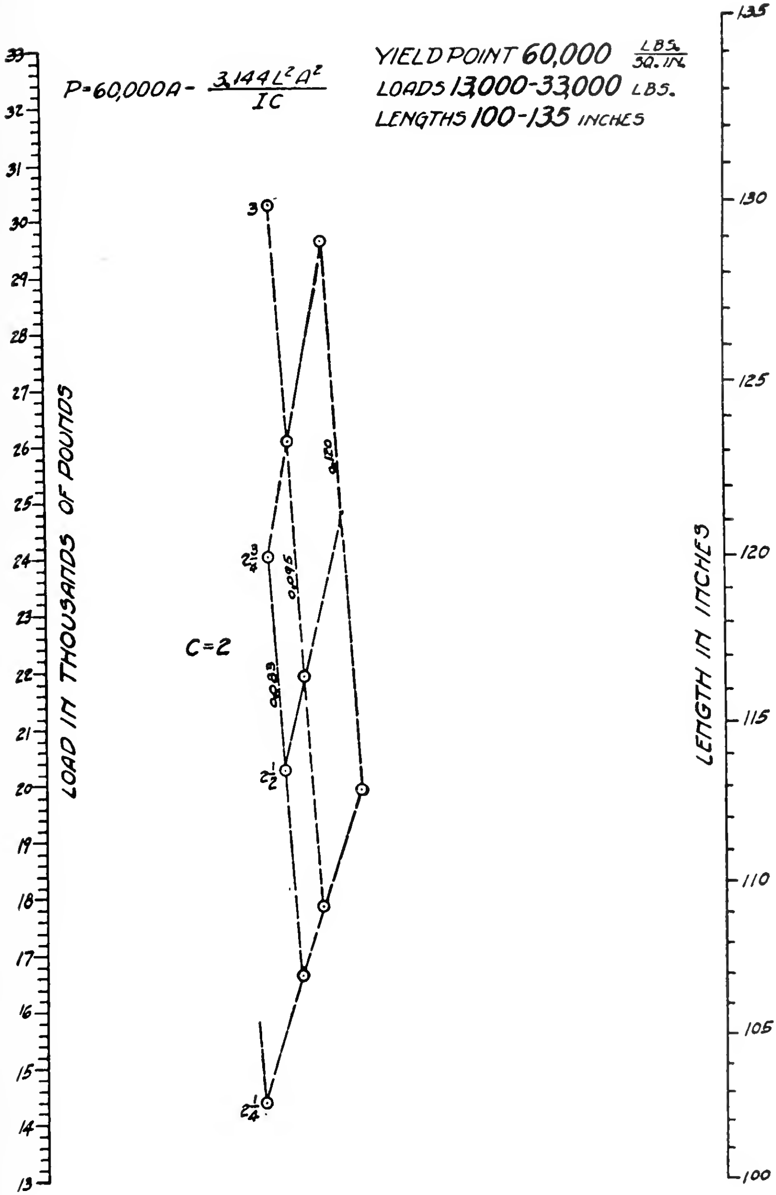
Nomogram for Steel Tubes, Short Column Range

FIG. 9 : 12



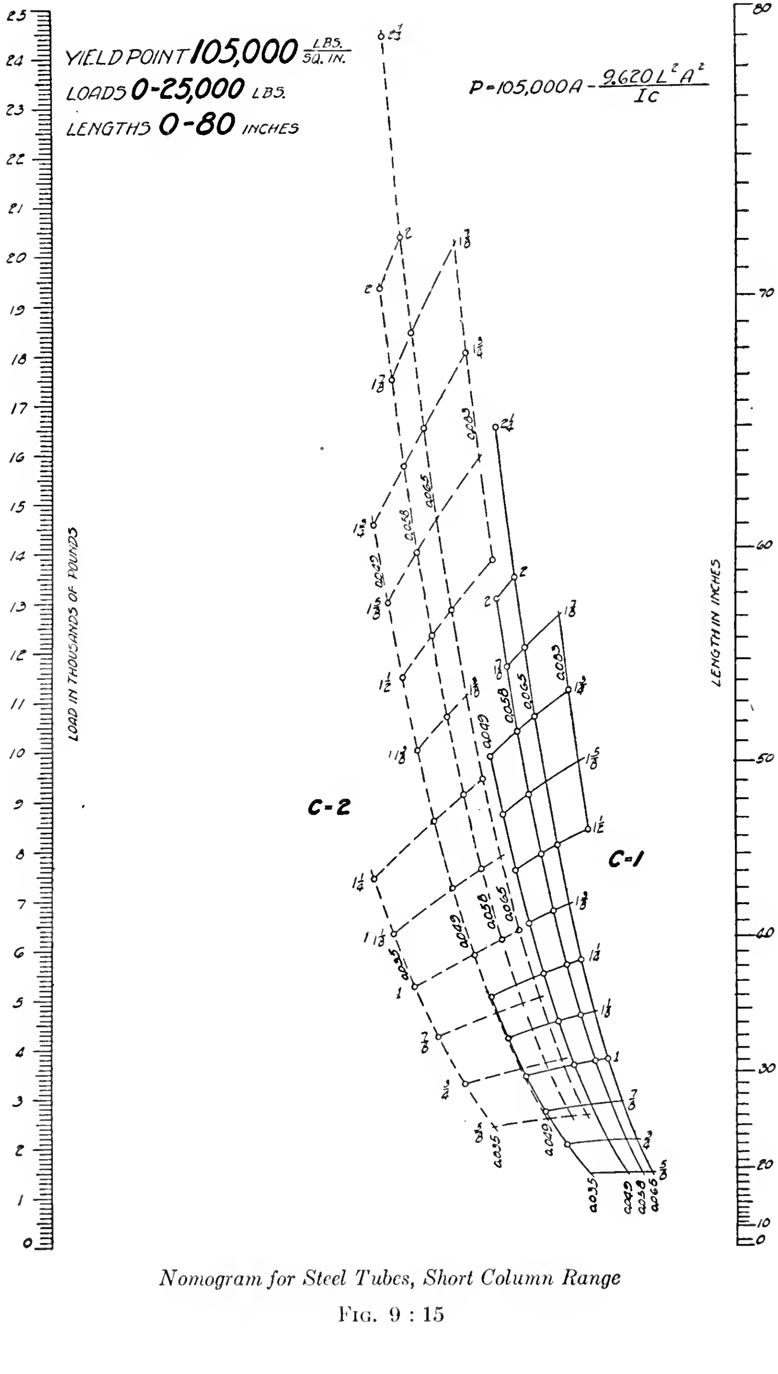
Nomogram for Steel Tubes, Short Column Range

FIG. 9 : 13



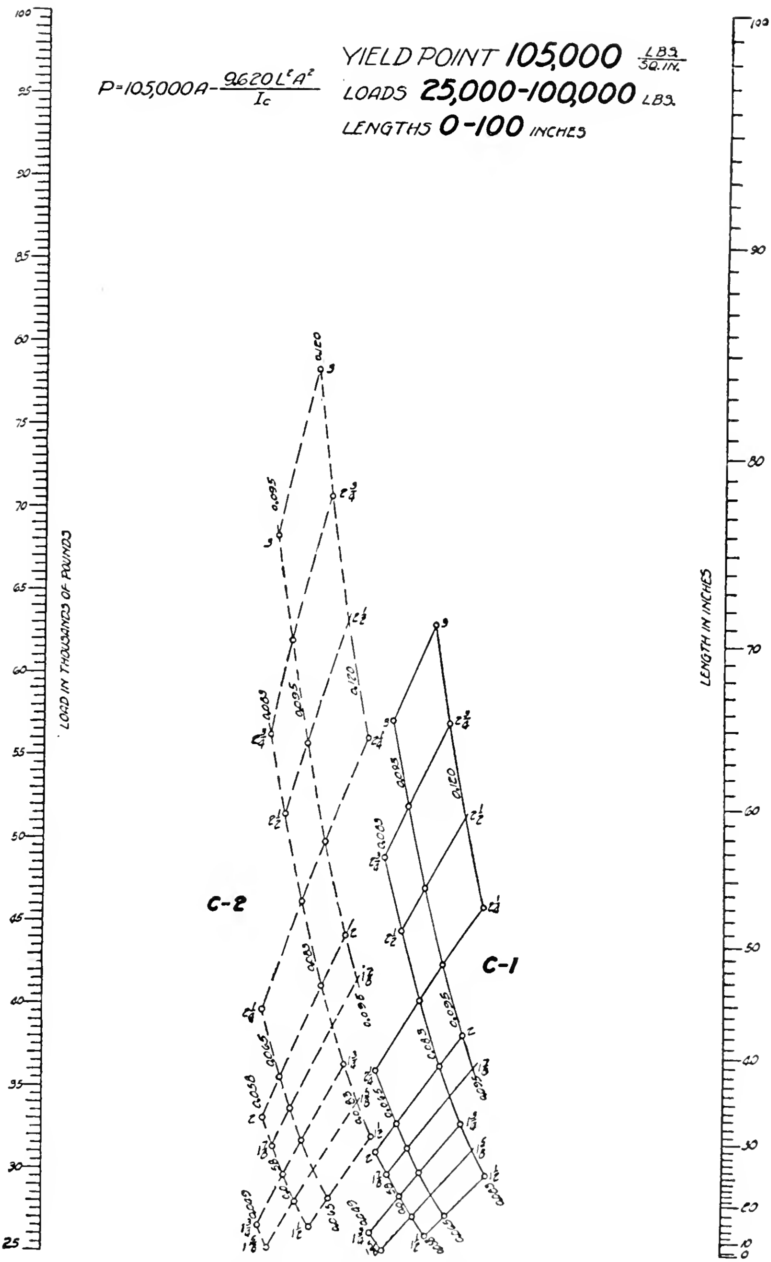
Nomogram for Steel Tubes, Short Column Range

FIG. 9 : 14



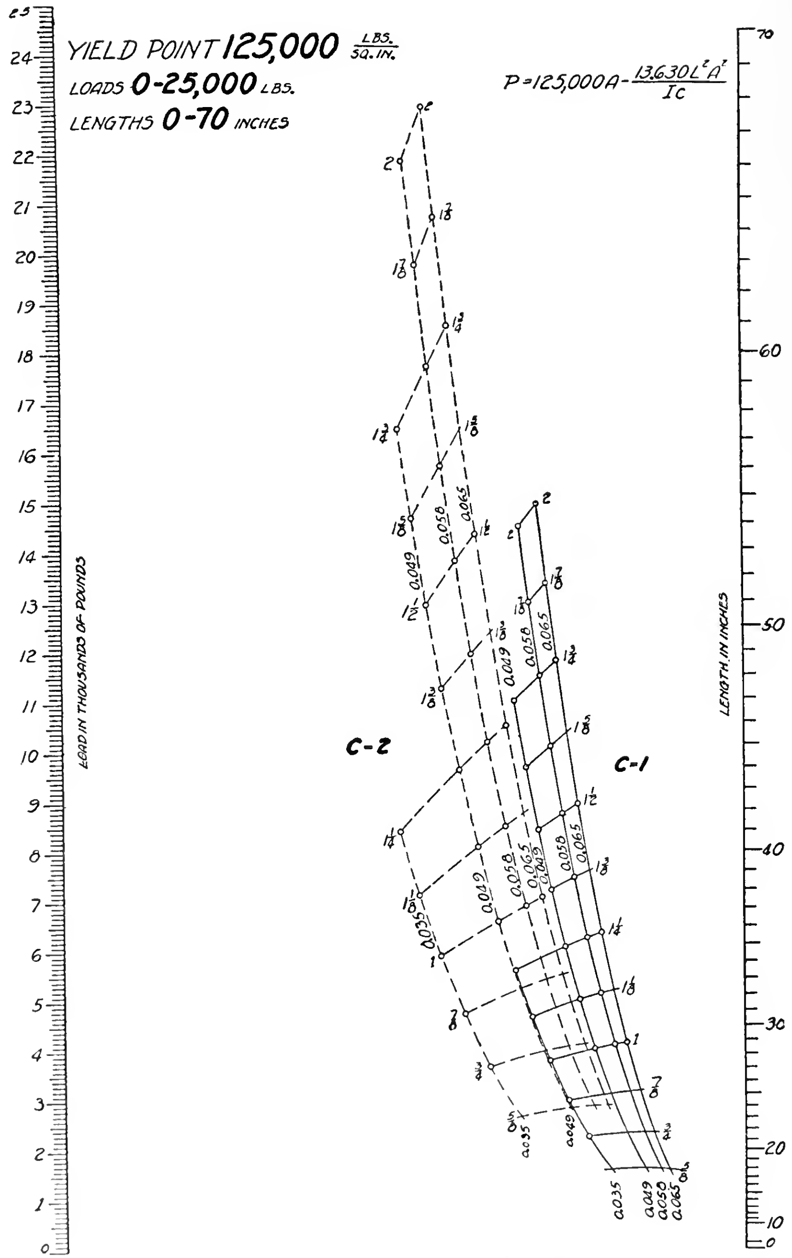
Nomogram for Steel Tubes, Short Column Range

Fig. 9 : 15



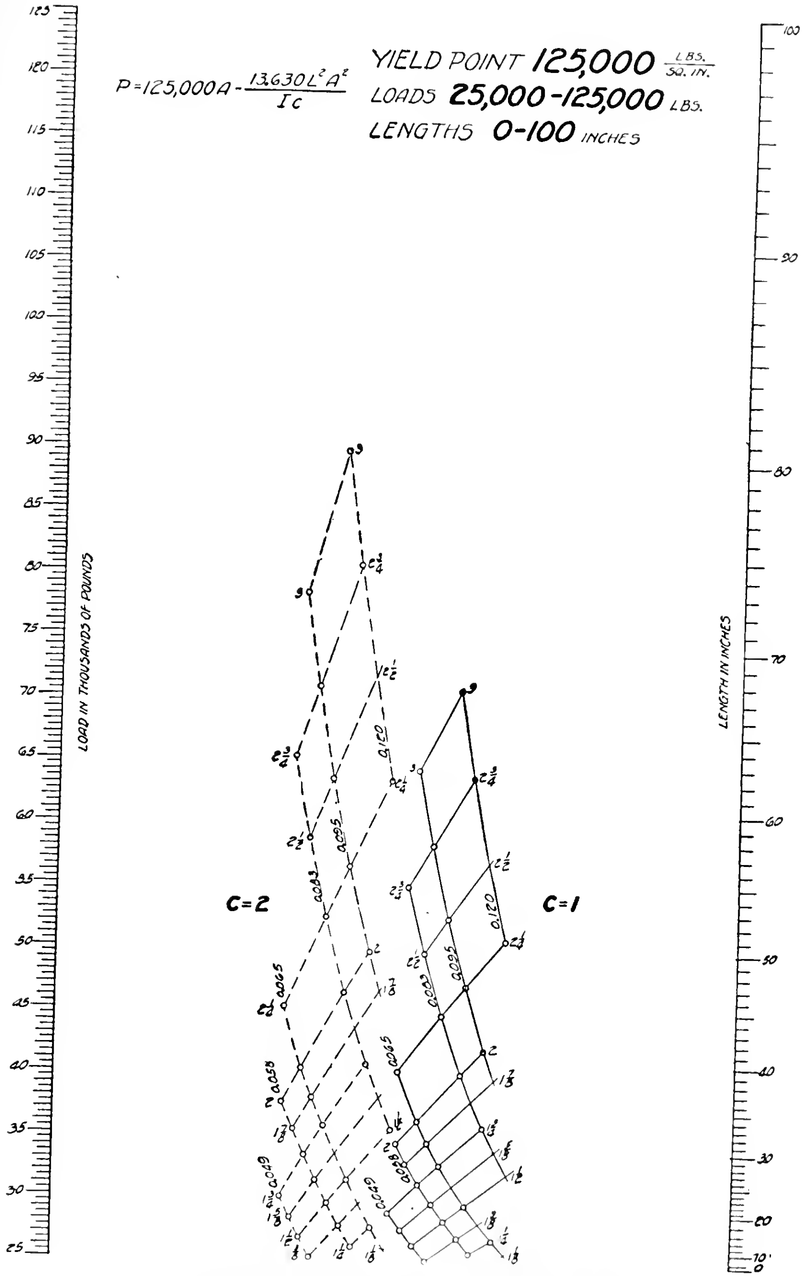
Nomogram for Steel Tubes, Short Column Range

FIG. 9 : 16



Nomogram for Steel Tubes, Short Column Range

FIG. 9 : 17



Nomogram for Steel Tubes, Short Column Range

FIG. 9 : 18

the parabolic formula applies. Figure 9 : 11 is for use with S. A. E. 1025 mild carbon steel tubing for which the yield point is assumed at 36,000 lb. per sq. in. and the modulus of elasticity at 28,000,000 lb. per sq. in. Figs. 9 : 12, 9 : 13, and 9 : 14 are for non-heat-treated chrome-molybdenum steel tubing with a yield point of 60,000 lb. per sq. in. and a modulus of elasticity of 29,000,000 lb. per sq. in. Figs. 9 : 15 and 9 : 16 are for alloy steel tubes heat-treated to a yield point of 105,000 lb. per sq. in., while Figs. 9 : 17 and 9 : 18 are for tubes heat-treated to a yield point of 125,000 lb. per sq. in.

In all these charts each tube is represented by two points, one for $c = 1$ and the other for $c = 2$. The points for $c = 1$ are to the right side of the chart and are connected with fine solid lines. Those for $c = 2$ are to the left and joined by broken lines. In each network of lines, one set represents outside diameters and the other wall thicknesses.[1] The intersections representing the tubes adopted as standard by the Army at the time the charts were drawn are designated by small circles. As both the outside diameter and wall thickness curves are smooth, fairly accurate extrapolations can be made to locate the points for tubes not covered by the charts as they stand.

A straight line joining a point on the length scale with a point representing a tube size, when extended, will intersect the load scale at the point representing the load that would be obtained from the Parabolic formula for the corresponding combination of length, tube size, and restraint coefficient. Conversely, if it is desired to determine what sizes of tubes would carry a given load over a given length, a line could be drawn between the corresponding points on the load and length scales. Any tube represented by a point above this line would carry the load, and the designer would naturally select the lightest of these unless there was some special reason for choosing another.

The problem the designer usually wishes to solve with these charts is what size of tube to use when the material, length, restraint coefficient, and design load are known. In many cases he will not know whether he should use the chart based on the Euler formula or one of those based on the Parabolic formula. To help settle this point, Table 9 : 8 has been compiled giving the "critical lengths" for different sizes of tubes, the critical lengths being the critical slenderness ratios of Table 9 : 7 multiplied by the radius of gyration of the tube. The critical length varies somewhat with the wall thickness as well as with the outside diameter, but this variation is so small that it is neglected in

[1] This system of connecting the points representing the various tube sizes was suggested by Walter L. Brock.

Table 9 : 8. This will cause no difficulty, however, since for lengths near the critical the two formulas give practically identical results.

In practice, if the designer is not sure whether to use the Euler or the Parabolic formula chart, he uses either and selects a tube. Then he looks on Table 9 : 8 to find out if the chart he used was the proper one for the tube he has chosen and the length in question. If not, he tries again on the other chart.

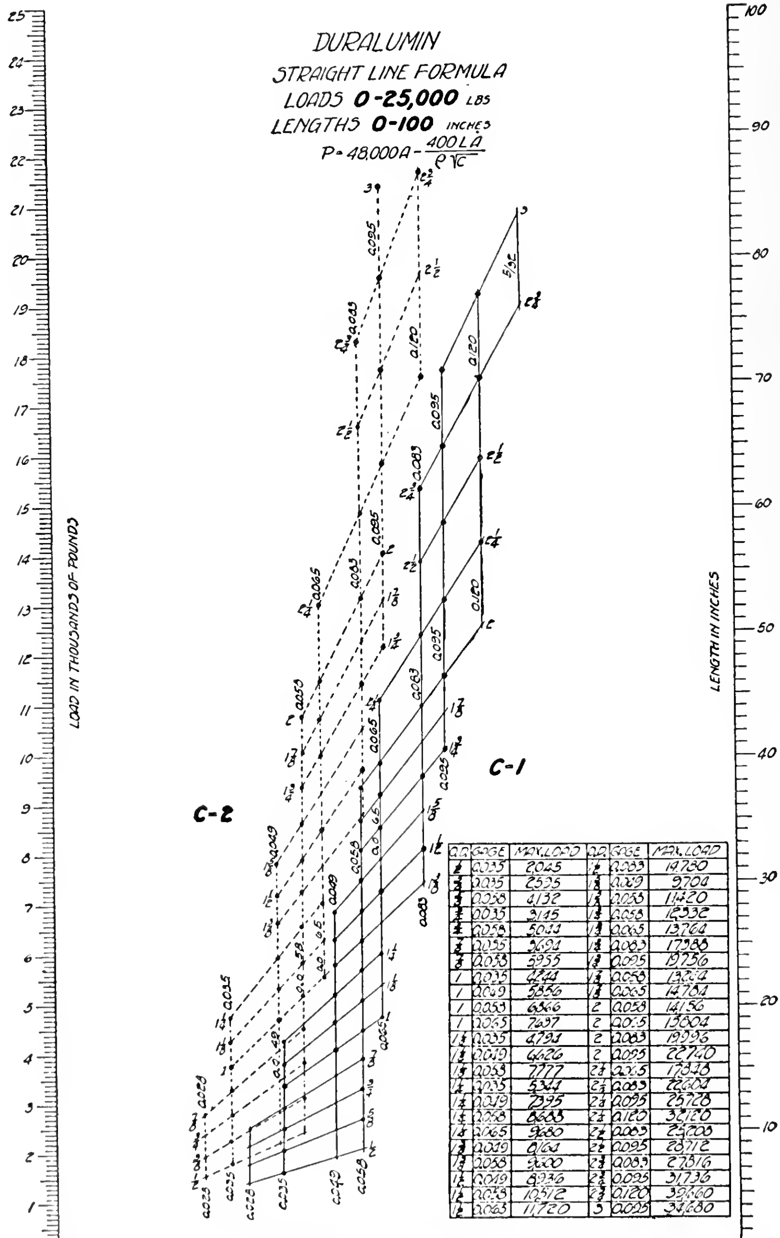
TABLE 9 : 8

CRITICAL LENGTH OF TUBING IN INCHES

Outside Diameter, Inches	Y.P.=36,000 E=28,000,000		Y.P.=60,000 E=29,000,000		Y.P.=105,000 E=29,000,000		Y.P.=125,000 E=29,000,000		Aluminum Alloy	
	$C=1$	$C=2$	$C=1$	$C=2$	$C=1$	$C=2$	$C=1$	$C=2$	$C=1$	$C=2$
1/4	10	14								
5/16	12	17								
3/8	15	21								
1/2	20	29								
5/8	26	37								
3/4	31	44	25	35	19	26	17	24	20	29
7/8	37	52	29	41	22	31	20	29	24	34
1	42	60	33	47	25	36	23	33	27	39
1 1/8	48	68	38	53	29	40	26	37	31	44
1 1/4	53	75	42	60	32	45	29	41	34	49
1 3/8	58	82	46	65	35	49	32	45	38	53
1 1/2	64	90	50	71	38	54	35	49	41	59
1 5/8	69	98	55	77	42	58	38	54	45	64
1 3/4			59	83	45	63	41	58	48	69
1 7/8			63	89	48	67	44	62	51	73
2			67	94	51	72	47	67	55	78
2 1/4			75	106	57	79	52	74	61	87
2 1/2			84	118	63	90	58	82	68	97
2 3/4			92	131	70	99	64	91	75	108
3			101	142	76	108	70	99	82	117

Figure 9 : 19 is a nomogram for solving the formula given in Table 9 : 7 for aluminum alloy tubes. It is used in the same way as the charts for the parabolic formula for steel tubes.

9 : 7. Numerical Examples — *First Example* — Determine the size required for the rear interplane strut of the wing cellule investigated in Article 7 : 6. Investigate round and streamline tubing of both non-heat-treated chrome-molybdenum steel and aluminum alloy, and also solid spruce streamline struts of a fineness of 4.0. This strut must be designed for the following conditions: axial load, 836 lb., restraint coefficient, 1.0, unsupported length, 56 in. This unsupported length



O.D.	GAGE	MAX. LOAD	O.D.	GAGE	MAX. LOAD
1/2	0.035	2045	1 1/2	0.083	14780
5/8	0.035	2595	1 5/8	0.049	9704
5/8	0.058	4132	1 5/8	0.058	11420
3/4	0.035	3145	1 3/4	0.058	12332
3/4	0.058	5044	1 3/4	0.065	13764
7/8	0.035	3694	1 3/4	0.083	17388
7/8	0.058	5955	1 3/4	0.095	19756
1	0.035	4244	1 7/8	0.058	13244
1	0.049	5856	1 7/8	0.065	14704
1	0.058	6866	2	0.058	14156
1	0.065	7637	2	0.065	15804
1 1/8	0.035	4794	2	0.083	19996
1 1/8	0.049	6626	2	0.095	22740
1 1/8	0.058	7777	2 1/4	0.065	17846
1 1/4	0.035	5344	2 1/4	0.083	22604
1 1/4	0.049	7395	2 1/4	0.095	25728
1 1/4	0.058	8688	2 1/4	0.120	32120
1 1/4	0.065	9680	2 1/2	0.083	25208
1 3/8	0.049	8164	2 1/2	0.095	28712
1 3/8	0.058	9600	2 3/4	0.083	27816
1 1/2	0.049	8936	2 3/4	0.095	31736
1 1/2	0.058	10512	2 3/4	0.120	39660
1 1/2	0.065	11720	3	0.095	34680

Nomogram for Aluminum Alloy Tubes, Short Column Range

FIG. 9 : 19

Note: Values in table give maximum tensile strength of tubes.

is less than the length of this strut, as listed on page 98, Art. 7 : 6, by 5.7 in., as it is assumed that the strut is connected to the spars by pins which, instead of being on the axes of the latter, are above the upper surface of the lower spar and below the lower surface of the upper. In this case 61.7 in. is the "theoretical" or "center-line" length of the strut and 56 in. its "true" length.

Since interplane struts are normally of the Euler class, Figs. 9 : 9 and 9 : 10 will be used. The steel tubes will be selected first. A line on Fig. 9 : 10 from 56 on the E = 29,000,000 side of the L scale to 836 on the load scale intersects the I scale at 0.0092 which is somewhat below the mark for the 1 × 0.035 round tube. That tube will therefore be satisfactory. If its margin of safety is desired, its strength must be found from the intersection of the load scale and the line from 56 on the length scale through the point representing the tube on the I scale. This is at 1117, so the margin of safety will be (1117 − 836)/836 = +0.336 or 33.6 per cent.

The points representing the moments of inertia of the streamline tubes are not marked on the I scale, so it is necessary to find a tube from Table 9 : 6 with I greater than 0.0092. The smallest such streamline tube is that made from a 1 3/8 × 0.035 round tube for which I = 0.01128. From the chart it can be determined that the strength of this tube would be 1017 lb. As the critical length for a 1-in. tube of the steel being used, and c = 1 as given in Table 9 : 8, is only 33 in., the use of the Euler formula chart is justified. The slenderness ratio of the streamline tube is 56/0.2767 = 202 which is much larger than the critical value of 98 given in Table 9 : 7, so this will also be an Euler strut.

In order to use the chart for aluminum alloy it is necessary to correct for the difference in modulus of elasticity by using the factor 2.7. As we wish to find an aluminum alloy strut to carry 836 lb., we use the chart to find the size of steel tube with E = 28,000,000 needed to carry 836 × 2.7 = 2260 lb. A line from 56 on the 28,000,000 side of the L scale to 2260 on the P scale intersects the I scale at 0.0257, which is below the mark for a 1 1/8 × 0.058 round tube, and still further below that for a 1 1/4 × 0.049 round tube. Tables 9 : 2 and 9 : 3 show that the latter is the lighter tube, having a smaller cross-sectional area, and Table 9 : 5 shows it to be stronger in bending, having a larger value of I/y, while its position on the I scale of Fig. 9 : 10 shows it to be the stronger as a column. This 1 1/4-in. tube would therefore be chosen in preference to the heavier but more slender and weaker 1 1/8-in. tube.

From Table 9 : 6 we find that the streamline tube made from a

1 5/8 × 0.049 round tube has a moment of inertia of 0.02571 so it is just large enough to carry the load. A check of the slenderness ratios and critical lengths of these aluminum alloy tubes shows that they are all in the Euler class and that their strengths as obtained from Fig. 9 : 10 in this manner are correct. Similar use of Fig. 9 : 9 shows that the streamline spruce strut needed would be one with a major axis of 4.25 in. and a minor axis of 1.0625 in. which would carry 918 lb.

It may have been noticed that the aluminum alloy struts called for were considerably larger than the corresponding steel struts, and it is of interest to compare their relative weights. This can be done by the use of Table 9 : 2 giving the weights per 100 in. of both steel and aluminum alloy tubes, and their areas multiplied by various unit stresses for the more commonly used sizes of round tubes. From this table it can be seen that 100 in. of the round steel tube selected weighs 3.01 lb. as compared to 1.87 lb. for round aluminum alloy. The weights of the streamline tubes are thus 4.13 lb. for steel and 2.46 for aluminum alloy. The 1 3/8 × 0.035 tube was not listed in the table so its weight was determined by proportion from that of the 1 3/8 × 0.049 size.

Second Example — Determine the size of round steel tube required for member *GL* of the truss shown in Fig. 8 : 11. Assume welded construction with $c = 2$ and steel with a yield point of 60,000 lb. per sq. in. As $c = 2$, if the tube selected is one in the Euler range, it will be one that with $c = 1$ would carry only 1140/2 = 570 lb. The length of this member is 48 in. and its design load 1140 lb. The line on Fig. 9 : 10 is therefore drawn from 48 on the L scale to 570 on the P scale, indicating the selection of a 3/4 × 0.035 tube, for which the critical length with $c = 2$ is only 35 in. According to the chart this tube, if $c = 1$, would carry 620 lb. Therefore with $c = 2$, it would carry 1240 lb., or 100 lb. more than the design load.

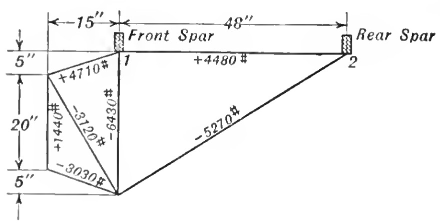


Fig. 9 : 20

Third Example — Determine the sizes of tubes and margins of safety for the members of the nacelle truss shown in Fig. 9 : 20. Assume all members subject to 75 per cent reversal of load, welded construction with $c = 2$, and the tubing to have a tensile strength of 55,000 lb. per sq. in. and a compression yield point of 36,000 lb. per sq. in. For this structure Fig. 9 : 11 would be used in practically the same manner as Fig. 9 : 10 in the first example above. The details of the solution therefore will not be gone into fully, but the computations and results

are presented in Table 9 : 9 in the manner in which such figures would be submitted in a report.

TABLE 9 : 9

DETERMINATION OF SIZES OF NACELLE TRUSS MEMBERS

Member	Load, Pounds	Length, Inches	Size Used	Strength, Pounds	M. S.
1–5	+4710	15.8	1×0.035	+5840	+0.24
	−3540			−3700	+0.045
1–2	+4480	48.0	1 1/4×0.035	+7350	+0.64
	−3360			−3820	+0.137
3–4	−3030	15.8	7/8×0.035	−3150	+0.040
2–3	−5270	56.5	1 3/8×0.049	−5610	+0.064
4–5	+1440	20 0		+2810	+0.95
	−1080		1/2×0.35	−1400	+0.296
3–5	−3120	29.2	1×0.035	−3340	+0.070
1–3	−6430	30.0	1 3/8×0.049	−6830	+0.062

Due to the requirement that the structure be designed to carry a 75 per cent reversal of load, compression as well as tension values are listed for those members for which the stress indicated in Fig. 9 : 20 is tension, and the margins of safety under these compression loads are computed. It may be noted that in every case, the compression margin of safety is lower than the tension margin although the tension load is the larger. No tension values are listed for the compression members as it should be obvious that the compression load is critical when it is the larger. The strengths in tension shown in Table 9 : 9 were read directly from Table 9 : 2.

The sizes shown are the smallest that would carry the design loads. Sometimes larger sizes are actually used for various reasons. Thus member *4–5* might be made considerably larger on the grounds that a 1/2-in. tube is too small to withstand the vibration to which it would be subjected when in such close proximity to the engine.

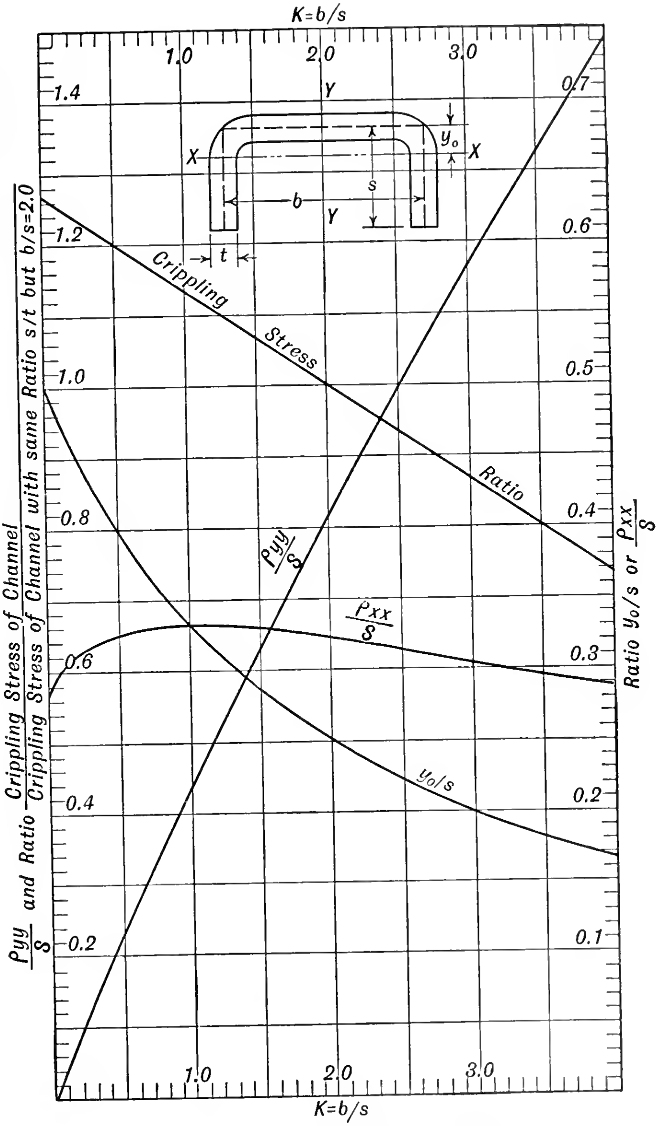
9 : 8. Column Shapes Subject to Local Failure — The column curves and formulas discussed in the preceding articles apply only to shapes that are not subject to local failure, or local buckling as it is sometimes called. If a piece of paper is rolled up into a tight cylinder it will carry a certain load as a column. If the same piece is wrapped so loosely that it forms a round tube with a wall thickness equal only to the thickness of the paper, and is loaded as a column, the area will remain unchanged, and the radius of gyration of the section will be greatly increased. According to the column formulas this would mean that the light tube of paper would carry more load than the closely

wrapped cylinder, but we know that it will not do so in fact. Under a very light load the walls of the thin tube will buckle locally and the whole will fail. The same effect would be obtained if instead of forming the sheet into a thin tube, it were formed into an angle or a channel. According to the column formulas it should carry much more load than when wrapped into a tight cylinder, but actually it will not carry nearly as much. This is an extreme case, but is a good example of the type of failure that is likely to take place in any column in which there are thin surfaces that are flat or where the ratio of the thickness of the material to the radius of curvature of the surface is very small.

The subject of local failure is one of great importance, particularly to the designer of metal airplanes, but lack of space precludes giving it more than a brief discussion here. It has been found that when the ratio of diameter to wall thickness of steel or aluminum alloy tubes is less than about 35 there is little danger of such failure, and that when such tubes are used as columns their strength can be computed on the basis of the formulas of Table 9 : 7. Nearly all tubes used in airplane design fall within this limitation of D/t ratio. In the few cases where a thinner tube is used, conservative practice is to test a tube of the size desired to make sure that it will carry the load.

There is little published information on the strengths of other shapes, most companies engaged in constructing metal airplanes either keeping the results of their tests confidential or else relying on guesses as to the strengths of the shapes they use. One shape in common use and on which reliable test data have been published is the aluminum alloy channel. Army Air Corps Information Circular 598, "Compressive Strength of Duralumin Channels" by Roy A. Miller, gives the results of tests run under his direction, and a method formulated by him for the design of such members. This method is outlined below. A designer wishing to use any shapes liable to local failure other than the type of channel covered by Mr. Miller's experiments should place no reliance on the general column formulas, but should determine the strength of his own shapes by testing. In fact, even when a channel of this type is used for an important structural member, it is desirable to check the strength of the size finally selected by test before it is actually used.

A channel such as is shown in Fig. 9 : 21 may be considered as made up of three flat sheets connected by small quadrants. If any one of these flat sheets is considered by itself its minimum radius of gyration will be only about 29 per cent of its thickness, t, and if its strength were computed by the Euler formula it would be very small, as the slender-



Characteristics of Aluminum Alloy Channels

FIG. 9 : 21

ness ratio would be very large. This, however, would be too severe an assumption, since one edge of each side is restrained by the back against deflecting in the direction of the small radius of gyration, while the back is similarly restrained along both edges. The amount of this restraint will depend partly on the average unit stress on the whole section and partly on its relative dimensions.

When the slenderness ratio of the whole section is large, the unit stress at the critical load is relatively small and the mutual support provided by the flat elements of the section will be sufficient to prevent local failure of the sides or back before the critical load of the section as a whole is reached. For such long channels, the tests indicated that the Euler formula could be used with the moment of inertia or radius of gyration of the entire cross-section.

As the slenderness ratio of the whole section decreases, its critical load increases. Eventually a point is reached where the mutual support of the elements of the section is insufficient to prevent local failure of one or more of them. The average unit stress in the channel when this local buckling takes place is not a function of the shape of the section alone, but for any given shape it decreases with increase in slenderness ratio of the channel as a whole. By testing several shapes with different slenderness ratios, and plotting the results, it was found that a combination of the Euler curve for aluminum alloy and a parabola could be drawn that would represent as well as could be expected the test results for each shape. The ordinate of the parabolic portion of this curve corresponding to zero slenderness ratio may be called the "crippling stress" for the shape for which the curve is drawn.

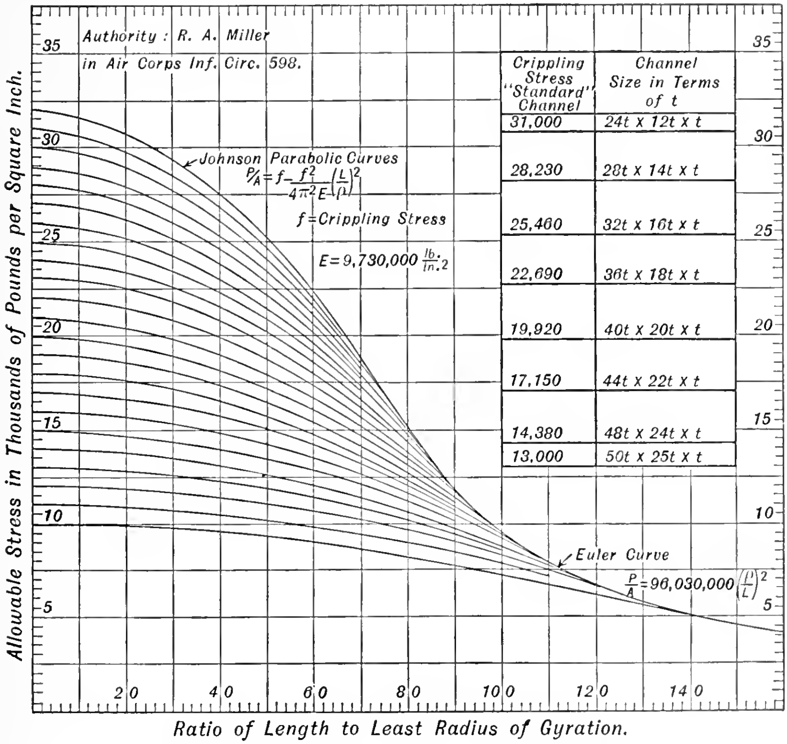
When the ratio of width of back to width of side of the channel is 2.0, and the width of side varies from $12\,t$ to $25\,t$, t being the thickness of the material, the crippling stress decreases uniformly with increase in width of side measured in terms of the thickness of the material. The crippling stress also decreases uniformly with increase in the ratio of width of back to width of side, the percentage decrease being practically independent of the ratio of width of the side to its thickness. The exact relationships between the crippling stresses of channels and the proportions of their cross-sections recommended for use in design are shown in Figs. 9 : 21 and 9 : 22 in connection with other curves that are useful in applying the method.

In determining the load that can be carried by a channel according to Miller's method, the first step is to determine the area and principal radii of gyration of the cross-section. If the channel is formed from flat stock it will have fillets at the corners as shown in Fig. 9 : 21. In computing the area, proper account must be taken of these fillets

and the true area will be

$$A = (2s + b - 0.8584\,r)\,t \qquad 9:5$$

where A = area of cross-section in square inches.
s = width of side in inches.
b = width of back in inches.
r = radius of fillet in inches.
t = thickness of sheet in inches.



Crippling Stress "Standard" Channel	Channel Size in Terms of t
31,000	24t x 12t x t
28,230	28t x 14t x t
25,460	32t x 16t x t
22,690	36t x 18t x t
19,920	40t x 20t x t
17,150	44t x 22t x t
14,380	48t x 24t x t
13,000	50t x 25t x t

Allowable Strengths for Aluminum Alloy Channels

FIG. 9 : 22

The dimensions s, b, and r used in this formula are those of the center line of the sheet and not the outside dimensions. In computing the distance to the centroid of the section from the center line of the back and the values of the radii of gyration, little error will be involved and much labor will be saved if the fillets are neglected and the quantities computed for a line of thickness, t. The formulas based on these

assumptions are: Distance from center line of back to centroid,

$$y_0 = \frac{s}{k + 2} \qquad 9:6$$

Radius of gyration about axis through centroid and parallel to back,

$$\rho_{xx} = \frac{s}{k + 2}\sqrt{\frac{2k + 1}{3}} \qquad 9:7$$

Radius of gyration about axis through centroid and parallel to sides,

$$\rho_{yy} = sk\sqrt{\frac{k + 6}{12(k + 2)}} \qquad 9:8$$

In these formulas k is the ratio b/s. Curves representing these formulas are plotted in Fig. 9 : 21.

Having found the geometrical properties of the section, the ratio of its crippling stress to that of a channel with $k = 2$ is found from the fourth curve of Fig. 9 : 21. The actual crippling stress can then be determined by multiplying this value by the crippling stress for a channel with $k = 2$ and the ratio of width to thickness of leg of the channel being investigated, as found from Fig. 9 : 22, interpolating if necessary. Fig. 9 : 22 also shows Parabolic-Euler column curves for different values of crippling stress between 10,000 and 32,000 lb. per sq. in. By using these curves, the allowable unit stress corresponding to the slenderness ratio and the crippling stress for the channel being investigated can be easily found. A numerical example of the application of this method is given below.

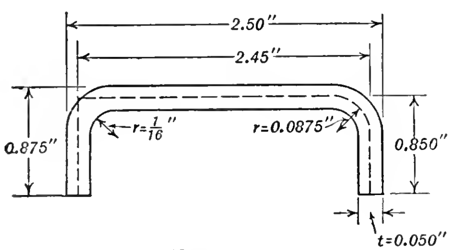


FIG. 9 : 23

9 : 9. Numerical Example—Determine the allowable axial compression for the duralumin channel shown in Fig. 9 : 23, for pin-ended conditions and a length of 10 in.

From Formula 9 : 5, the area of the section is

$$A = (2 \times 0.85 + 2.45 - 0.0875 \times 0.8584) \times 0.05 = 0.2037 \text{ sq. in.}$$
$$k = 2.45/0.85 = 2.88$$

From Fig. 9 : 21 or Formulas 9 : 7 and 9 : 8,

$$\rho_{xx} = 0.85 \times 0.308 = 0.262 \text{ in.}$$
$$\rho_{yy} = 0.85 \times 1.121 = 0.953 \text{ in.}$$

As $L = 10$ in., the maximum slenderness ratio is $10/0.262 = 38.2$. The ratio of width to thickness of side is $0.85/0.05 = 17$. From Fig. 9 : 21 the ratio of crippling stresses for $k = 2.88$ is 0.882. From Fig. 9 : 22 the crippling stress for a channel $34\,t \times 17\,t \times t$ is 24,080 lb. per sq. in. Therefore the crippling stress for the channel being investigated is $0.882 \times 24{,}080 = 21{,}240$ lb. per sq. in.

From Fig. 9 : 22 using $L/\rho = 38.2$ and interpolating between the curves for crippling stresses of 21,000 and 22,000 lb. per sq. in., the allowable stress is 19,500 lb. per sq. in. Multiplying this figure by the area, 0.2037 sq. in., the allowable load is found to be 3970 lb.

9 : 10. Plate Girder Web Stiffeners — From Mechanics we know that when shear stresses exist at any point on the cross-section of a beam tensile and compressive stresses also exist on planes inclined with respect to that cross-section, the degree of inclination depending on the magnitude of the direct stresses existing at that point at the same time as the shear. At the neutral axis of a beam, where the direct stresses are zero, the planes of maximum tensile and compressive stress, which are at right angles to each other, lie at an angle of 45 degrees to the plane of the shear. Hence the web of a beam such as a plate girder, being designed to carry the shear acting on the beam, must carry compressive forces as well. That such forces actually exist is proved by the failure of plate girder and similar webs by buckling or crinkling. To counteract such failures, angles or other stiffeners may be provided at intervals along a single web plate, or internal bulkheads may be used in box beams.

The spacing of such stiffeners presents a problem for which no accurate theory has yet been developed; the following method, which is an adaptation of standard bridge practice, has been used on metal airplane spars and, as indicated by a few tests, gives reasonable and satisfactory results.

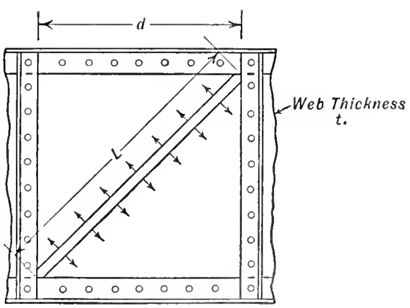


Fig. 9 : 24

Assuming that the bending moment carried by the web is small at sections where the shear is a maximum, the compressive stresses on the web act at an angle of approximately 45 degrees throughout its entire depth. Then, considering a strip of unit width as shown in Fig. 9 : 24, we have a column of thickness, t, length L, and width, 1.0. The radius of gyration of such a column, assuming it to buckle laterally,

would be $\frac{t}{\sqrt{12}}$ and if d is the distance between stiffener angles, $L = d\sqrt{2}$, then $\frac{L}{\rho} = \frac{d\sqrt{2}\sqrt{12}}{t} = 4.9\frac{d}{t}$ or very nearly $5\frac{d}{t}$. Were the column to be pin-ended the allowable stress could be found from the approximate formulas of Table 9 : 7 with $c = 1.0$; but it is to be noted that the ends are restrained to some extent by the flange angles and, in addition, the whole strip is materially restrained against lateral failure by the diagonal web tension acting at an angle of 90 degrees to the web compression. A coefficient considerably in excess of 1.0 can therefore be used. Following bridge practice and assuming an aluminum alloy web, we can reduce the coefficient of the second term of the straight line column formula from 400 to 150, giving $P/A = 48{,}000 - 150\,L/\rho$ or $P/A = 48{,}000 - 750\frac{d}{t}$ for aluminum alloy web plates.

From Mechanics we also know that the intensity of the compressive stress is equal to the intensity of the shear for the condition of maximum compression on a plane at an angle of 45 degrees with the axis. Hence $P/A = f_s = 48{,}000 - 750\frac{d}{t}$. But $f_s = \frac{SQ}{bI}$ which, for a solid rectangular plate gives $f_s = \frac{3}{2}\frac{S}{A}$ and for a rectangular plate containing rivet holes which must be deducted when determining Q and I, $f_s = \frac{4}{3}\frac{S}{A}$ approximately. Then $d = \frac{t}{750}\left(48{,}000 - \frac{4}{3}\frac{S}{A}\right)$ where S is the shear on and A is the gross area of a vertical section of the spar web. If we limit the maximum L/ρ of a strip of this sort which has no stiffeners to 175, this being reasonable for aluminum alloy, we have a maximum ratio of $d/t = 35$ for a web plate having no stiffeners. For steel, with its higher modulus of elasticity, the maximum L/ρ would be 300 giving a limiting ratio of $d/t = 60$ where no stiffeners are used.

Similar expressions may be developed for other materials but will not be given here.

The outstanding leg of web stiffener angles should be from one-fifteenth to one-twentieth of the length of the angle to give the desired stiffness. Exact figures cannot be given but it is believed the above will be satisfactory. When the stiffener angles are used to distribute concentrated loads over the web, as at points of support or points where other large concentrated loads are applied, the angles should be designed as pin-ended columns whose length is equal to half the actual depth of the girder. In addition, the area of the outstanding leg should be sufficient to carry the entire load being distributed without exceeding the crushing strength of the material, and sufficient rivets should be used to transmit this load from the stiffener angle into the web plate.

PROBLEMS

9 : 1. Determine the sizes of all wires in the drag truss for which the stresses are shown in Fig. 7 : 14. Assume round swaged tie rods and use sizes large enough to give a margin of safety of at least 30 per cent.

9 : 2. Design the lift wires for the rear truss of the airplane analyzed in Art. 7 : 6. Use two wires of streamline section, each of which must have sufficient strength to carry 65 per cent of the design load.

9 : 3. Design the compression ribs for the drag truss shown in Fig. 7 : 14. Assume the compression elements to be spruce members of rectangular cross-section, one element being glued to each side of the 1/8 in. thick plywood web of the compression rib in such a way that the long axis of each element intersects the spars at mid-height. Due to the stiffness of the web it may be assumed that failure will occur in a plane normal to the plane of the web. Assume a fixity coefficient of 1.0.

9 : 4. Design compression struts of round steel tube for the drag truss shown in Fig. 7 : 14. Assume a pin joint at each end and investigate mild steel and chrome-molybdenum steel tubes.

CHAPTER X

CONNECTIONS

In the previous chapters consideration has been given to the analysis and design of individual members subjected to bending, torsion, shear, or axial stresses, or a combination of such stresses, but no attention has been paid to the methods of attaching these members to one another to form a complete structure. Of the three main types of connections commonly used in airplanes there is but one, that classified as "mechanical," which is susceptible to reasonably accurate methods of analysis. For this reason fittings and connections are a source of much trouble in airplane structures and it is never desirable to design such details with small margins of safety.

The main types of connection may be classified as those depending on fusion, wherein the parts to be joined are melted while in contact so that, when cooled and hardened, they become a single piece; those depending on the adhesive action of some substance such as glue, solder, or brazing spelter when placed between the parts to be connected; and those depending on mechanical connection of the members wherein local compression due to bearing is produced in the members at the points where they are keyed, grooved or threaded, or where they are pierced by bolts, rivets or pins. Of the secondary types of joints some depend primarily on friction between the parts such as is developed by clamping, lashing or splicing. Others, such as sewing and nailing, depend partly on friction, partly on mechanical bearing. The first three types are of equal importance from the standpoint of use in an airplane, the secondary types are of limited use because of their low efficiency.

10 : 1. Fused Joints — The typical method of connecting airplane parts by fusion is by welding. Electric arc and gas welding and, to a less extent, resistance welding are employed, their use being limited, practically, to steels which are not adversely affected by the heat. To weld a joint involves the application of sufficient heat to bring the materials to a plastic or molten state and hence coarsens the grain and vitiates the effect of any heat-treatment or cold-working which may previously have been given the materials. The molten metal at the joint becomes, when cooled, essentially a cast metal and the material adjacent to the weld receives sufficient heat to be annealed.

Plain carbon steels and the chrome-molybdenum steel alloy commonly

used in aircraft suffer very little loss in tensile strength due to welding, the strength of the cast steel at the weld being but little less than that of the original wrought material. Some loss is sustained when cold-rolled sheet or cold-drawn tube is welded because the refinement of grain produced by the cold-working is lost near the weld. In any case a welded joint is less ductile and has a lower resistance to impact than the original material, so that such joints suffer a loss of strength in bending and shock. It is therefore customary to assume the efficiency of welded joints at 80 per cent of the strength of the original steel in its normalized condition and, since welded joints are more reliable under direct compression and shear due to bending than under direct tension, tension due to bending, or shear due to torsion, it is standard practice to detail welded connections so that the weld is as far as possible under direct compression or shear.

The U. S. Army Air Corps in its Handbook of Instructions for Airplane Designers states that no splices shall be made in main structural members by butt welds subjected to direct tension or bending stresses unless the joint is reinforced by riveting, pinning, telescoping or equivalent auxiliary means. This is excellent practice.

Heat-treatment after welding refines the grain and alters the structure of the steel so that the properties of the joint are improved. Alloy steels may therefore be welded and subsequently heat-treated if the assembly involved is not too large. It is considered good practice to assume the strength of such a joint after heat-treatment to be only 80 per cent of that which would be obtained with the same material if it were not welded.

Heat-treated aluminum alloys cannot be welded satisfactorily since the effect of the heat-treatment, upon which these alloys depend for their strength, is lost during the welding operation, and the cast structure of the material at the joint, besides being extremely brittle, has very low strength properties. Plain aluminum may be welded satisfactorily, but welding of its high-strength alloys is generally prohibited in the primary members of an airplane structure.

In joining two members by welding it is desirable to have them of as nearly the same thickness as possible so that one will not be overheated and burned while the other is being brought to the required plastic or molten condition. Steel sheets or tubes whose gage is less than 0.035 in. are extremely difficult to weld without burning out a part of the carbon, and their use in welded structures is not considered good practice on this account.

With the gages of materials used in aircraft a well-made weld will have a fillet of sufficient size to develop at least 80 per cent of the

strength of the original material so that it is, generally speaking, unnecessary to investigate such joints on the basis of the weld carrying a definite number of pounds per linear inch. When any doubt is felt as to the strength or dependability of a joint, it may be reinforced by the addition of gusset plates or, in the case of a splice in a tension or compression member, by the use of a "fish mouth" such as is shown in Fig. 10 : 1. Tests indicate that a concentric butt weld such as is shown in Fig. 10 : 2 is satisfactory but that if it is to withstand vibration, as for instance in the case of engine mounts, it is well to reinforce the joint by the use of gussets as shown in Fig. 10 : 3. Eccentric joints such as that in Fig. 10 : 4 are not dependable and should be avoided. If they must be used, as is often the case when making a field repair, they should be reinforced by gussets. Due to the fact that the center lines of the members do not intersect at a common point, bending is produced in the joint and the weld has a tendency to strip.

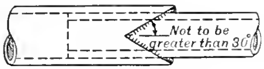


FIG. 10 : 1

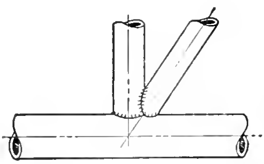


FIG. 10 : 2

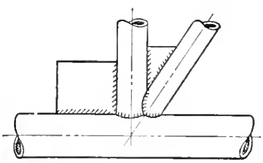


FIG. 10 : 3

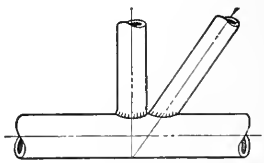


FIG. 10 : 4

10 : 2. Brazing — The method of adhesion most used for connecting steel parts is brazing, in which the adhesive material is a copper-zinc mixture inserted between the parts by melting the mixture with an air-gas flame or by dipping the whole assembly into a tank containing the molten spelter. For the best results the mixture should be drawn into the space between the members by capillarity. The strength of the joint depends on the surface areas of contact, the clearance between the parts joined, and, above all, the skill of the workman who makes the joint.

Although the brazing mixture may have a shearing strength of 40,000 lb. per sq. in., this cannot be relied upon in practice. A shearing strength of 20,000 lb. per sq. in. is reasonably dependable and is recommended for use where conditions for brazing are most favorable. When a brazed joint carries tension due to bending, a stress of not more than 10,000 lb. per sq. in. should be used, as there is a tendency present to strip the joint and, due to oxidation, the mixture is weakest at the edges where the unit stress is greatest.

Dip brazing in which the parts to be joined are assembled and then dipped into the molten brazing mixture is more dependable than brazing with an open flame. In any case the work must be carefully done and should conform to some reliable process specification such as that promulgated by the Navy Department.

No type of welding operation should be performed on a part or assembly that has previously been brazed as the brazing metal permeates the steel adjacent to the weld and causes a serious decrease in ductility. Plain carbon, chrome-molybdenum, and chrome-vanadium sheet and tubing may be brazed satisfactorily. Nickel steels give trouble as they are brittle at brazing temperatures and liable to crack. If the brazing mixture is properly chosen — 79 to 82 per cent copper and 21 to 18 per cent zinc is about right — the joint may be heat-treated after brazing so long as the quenching temperature is not higher than 1650 degrees Fahrenheit.

10 : 3. Soldering — Soldering is the application of a tin-base or silver-base alloy to two surfaces to form a joint. The alloy generally used for "soft" solder is 50 per cent tin and 50 per cent lead, a mixture which has a strength of about 6,000 lb. per sq. in. in shear. Soldered joints, however, have no proportional limit and the steady application of even small loads will cause such joints to slip and gradually fail although the breaking strength under a quickly applied load may be close to the ultimate strength of the solder. This characteristic of solder prevents its use in important joints except where it is used simply to prevent relative movement of parts as in the case of wrapped terminals where the real load is taken care of in some other manner.

10 : 4. Glueing — The method of adhesion used to connect wood parts is glueing, and it is almost the only method used to connect wood to wood in airplane construction. The glue most commonly used is made from casein — obtained from milk — combined with lime and chemical solvents, usually by the manufacturer's patented formula. The better grades of animal and blood albumin glues are often used in the manufacture of plywood and propellers. Vegetable glues are not satisfactory on account of the difficulty of spreading by hand and their lack of water resistant qualities. Prepared liquid glues are not sufficiently uniform to be dependable, have very low resistance to moisture and, generally, low strengths.

The strength of glued joints depends very largely on the manner in which the glue is used. A very thin or a very thick joint of animal glue is likely to be weak. The thin joints, called starved joints, can be readily caused by having the glue too hot, having the parts to be glued heated too much, or having the workroom too warm. One of

the great advantages of casein glue is that it can be mixed with cold water and does not require a heated room or heated wood for its successful use. This advantage, which enhances its dependability, offsets the fact that it is not quite so strong as the best animal glue. In most cases casein glues have sufficient strength to develop the full shearing strength of the wood to which they are applied, a value of 2800 lb. per sq. in. being developed on a glued joint carrying shear between maple blocks and about 2000 lb. per sq. in. being developed in the case of spruce.

The strength of a glued joint depends largely on the angle between the grain of the pieces joined. The values given above are for members whose grain runs in the same direction. When the angle between the grain of the two pieces is 90 degrees the shearing strength of the joint is reduced about 50 per cent, while the decrease in strength for an angle of 45 degrees is about 40 per cent.

The strength of a glued joint in tension is extremely variable and such joints should never be used in an airplane structure. They have a tendency to open up and permit moisture to enter so that they deteriorate rapidly. Glued joints are ordinarily designed so that they will transmit the load from one member to another by shear across the joint.

The glued joints in wing beams are always arranged so that this is the case. The simplest type of joint in such members, from the standpoint of analysis, is that used to attach filler blocks to the chords and webs of the spars at points of fitting attachment. Fig. 10 : 5 shows such a fitting carrying an end load from the chords of a wing spar to the fuselage or some similar structure. It is obvious that the stress per square inch on the glue is equal to the end load, P, divided

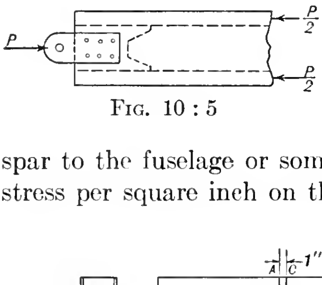


FIG. 10 : 5

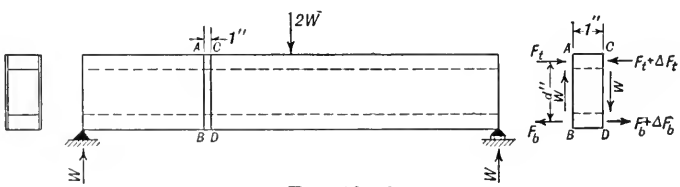


FIG. 10 : 6

by the area of the filler block in contact with the two chords, the stress developed on the glue being pure shear.

In the case of a built-up box beam subjected to simple bending, such as that shown in Fig. 10 : 6, however, the stress developed between the plywood webs and the spruce chords is due to longitudinal shear.

The load is applied to the beam from the ribs in a direction approximately normal to the longitudinal axis of the beam and so produces shear in the webs. If we isolate a strip one inch long from a beam loaded as shown in Fig. 10 : 6 we note that the forces acting on the strip consist of two equal and opposite vertical loads, representing the shearing forces in the webs, and two resultant horizontal forces, ΔF_t and ΔF_b, which act on the chords of the beam. In order to have equilibrium on the strip the moment produced by the shearing forces, W, acting on sections one inch apart must be equal to the moment produced by the horizontal forces, ΔF_t and ΔF_b, which act with an arm of d inches. Also, since $\Sigma H = 0$ it follows that $\Delta F_t = \Delta F_b$. Hence, if we let d be the distance between the centroids of the two chords and take moments about the point of intersection of ΔF_b and the section CD we have, $W \times 1 - \Delta F_t \times d = 0$, whence $\Delta F_t = \dfrac{W}{d}\cdot$ But $\Delta F_t = \Delta F_b$ and W is the transverse shear on the strip; so we find that the longitudinal shearing force acting on a 1-in. length of the joint between the web and chord of a box beam is equal to the transverse shear acting on the strip divided by the distance between the centroids of the chord members.

This relationship is approximate in that it assumes the forces ΔF_t and ΔF_b to be applied at the centroids of the chords or, in other words, that the distribution of stress over each chord is uniform. The error involved is, however, small in the case of deep beams having thin chord members and it is not until the depth of the spar is decreased beyond that ordinarily occurring in airplane practice that the error, which is always on the conservative side, becomes appreciable. A further approximation is entailed in neglecting the effect of a load applied between the ends of the strip such as would occur with a uniformly distributed load of w lb. per in. on the beam. Provision may, however, be made for such a load by noting that ΔF_b is the horizontal component of the force on the joint at the bottom chord; so, by assuming $\dfrac{w}{2}$ to be the vertical component, the resultant force to be carried by the glue would be the square root of the sum of the squares of these components.

For a more precise determination of the shearing force between web and chord the formula for longitudinal shear given in Chapter V may be used. In that chapter it was noted that the longitudinal shear stress, f_s, equalled $\dfrac{SQ}{bI}$ where S was the tranverse shear at the section under consideration, Q was the static moment, taken about the neutral axis, of the part of the cross-section above the section under considera-

tion and I was the moment of inertia of the entire cross-section about its centroid. Considering the cross-section shown in Fig. 10 : 7, we note that had the section been a solid rectangle 1.25 in. wide and 6.5 in. deep we could get the longitudinal shear stress at section BD by using the above formula. In that case b would equal 1.25 in., the width of the beam. But the beam is not solid and the longitudinal shear which would have occurred on BD is transmitted into the web by the glue joints on sections AB and CD. One-half the shearing force would then be carried by each of the two joints instead of by the section BD. Hence, b would be 1.0 in. instead of 1.25 in. and S would be one-half the transverse shear at the section.

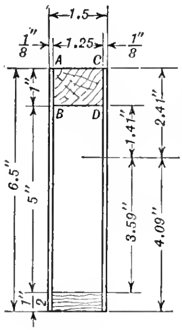


Fig. 10 : 7

10 : 5. Illustrative Problem — Given a box spar having the dimensions shown in Fig. 10 : 7, they being the same as those used in the problem in Chapter V. At the critical section the vertical shear is 1089 lb. What is the intensity of stress on the glued joint AB?

By the approximate method:

$$S = 1089 \text{ lb.}$$

$$d = 2.41 - 0.50 + 4.09 - 0.25 = 5.75 \text{ in.}$$

$$F_t = \frac{1089}{5.75} = 189 \text{ lb.}$$ which represents the increment in the load in the top chord per inch of span. As there are two joints, AB and CD, carrying this load the stress on each becomes

$$\frac{189}{2.0 \times 1.0 \times 1.0} = 94.5 \text{ lb. per sq. in.}$$

By the "exact" method:

$S = 1089$ lb. as above.

$Q = 1.0 \times 1.25 \times 1.91 = 2.39 \text{ in.}^3$

$I = 13.90 \text{ in.},^4$ from page 71.

$b = 1.00$ in. for one joint, but there are two to carry the load.

$$s = \frac{1089 \times 2.39}{2 \times 1 \times 13.90} = 93.5 \text{ lb. per sq. in.}$$

The allowable stress on the glue joint between a spruce-plywood web and spruce chord is 250 lb. per sq. in.

The agreement between these values is excellent and indicates the much less laborious "approximate" method to be quite precise when applied to a representative spar. Attention is called to the fact that

the stress on the glued joint on the lower chord would be greater than that on the upper since the depth of the joint is less.

10 : 6. Rivets, Bolts and Pins — Joints made by piercing the members to be connected and inserting auxiliary members such as bolts, rivets or pins are of frequent occurrence in airplane structures. Fig. 10 : 8 shows how the stresses are transmitted from one main member to the other by producing shear and bending in the auxiliary member and relatively severe local compressive stresses at the points where the main members bear on the auxiliary.

The force P in the thinner of the members shown in Fig. 10 : 8 is transmitted to the rivet by bearing of the plate on the upper half of the rivet over the distance AB whence it is carried by the rivet and transmitted into the thicker plate by bearing between the lower half of the rivet and that plate. Since the two forces, P, do not act along the same line they produce shear and bending in the rivet, the bending generally being small enough to be neglected when the joint is made, as it is here, with no intermediate members between the two main ones.

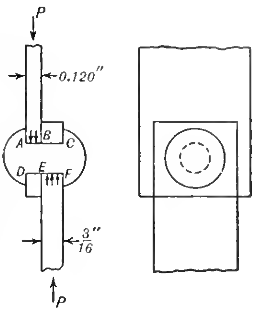
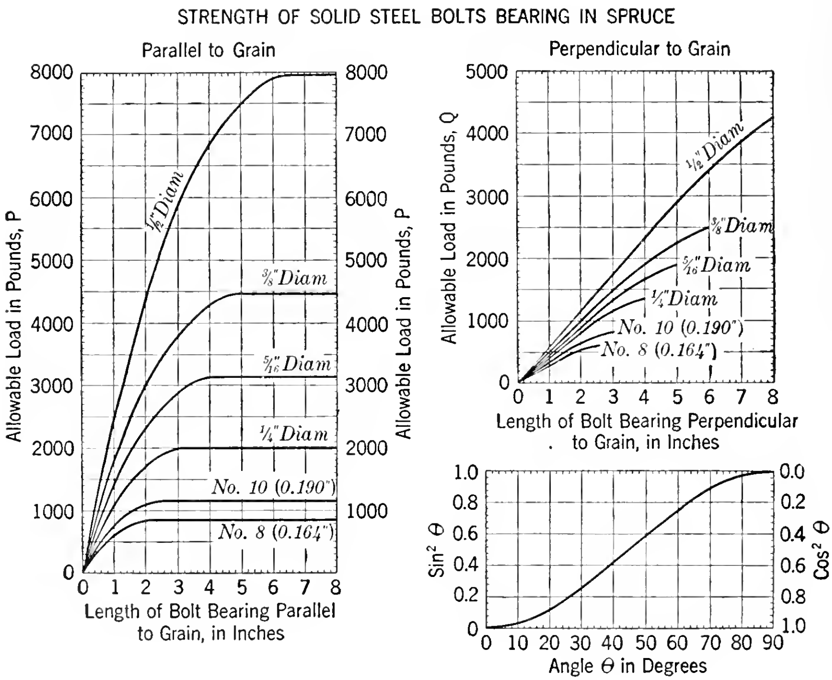


FIG. 10 : 8

Although the exact distribution of stress around a rivet hole has been shown by photoelastic analyses to be otherwise, it is assumed in practice that the bearing stresses are uniform across the diameter of a rivet. The effective bearing area is then equal to the thickness of the loaded plate times the diameter of the rivet. In addition to bearing, a rivet must also be investigated as to its ability to carry the shear across a section such as BE. Table 10 : 1 gives the strength of rivets, bolts and pins of various diameters and various materials in shear. Table 10 : 2 shows the allowable bearing strength of rivets or bolts on mild steel sheet of different thicknesses. The following tables, 10 : 3 to 10 : 5, and Fig. 10 : 9 give the same data for other materials.

Assuming the load P in Fig. 10 : 8 to be 2000 lb. and the shank of the rivet 1/4 in. in diameter, the bearing produced on the thinner plate by the rivet would be $\frac{2000}{1/4 \times 0.120} = 66{,}700$ lb. per sq. in. The allowable stress intensity on cold-rolled sheet is 95,000 lb. per sq. in. and, as shown in Table 10 : 2, the allowable bearing load on a 0.120-in. plate for a 1/4-in. rivet is 2850 lb. From Table 10 : 1 the allowable shearing strength of a steel rivet 1/4 in. in diameter is 1719 lb., which is not sufficient to carry the 2000 lb. load. The strength of this joint is therefore limited by the strength of the rivet in shear.



Strength of Bolted Joints in Spruce[1]

FIG. 10 : 9

[1] Based on data in N. A. C. A. Technical Note No. 296, "Bearing Strength of Wood under Steel Aircraft Bolts and Washers," by G. W. Trayer of the Forest Products Laboratory, Madison, Wis.

TABLE 10 : 1

SHEARING STRENGTH OF RIVETS, BOLTS OR PINS

Size of Rivets, Bolts or Pins	Strength in Single Shear			
	Allowable Shear			
	30,000 lb. per sq. in.[1] Aluminum Alloy	35,000 lb. per sq. in. 55,000[2]	65,000 lb. per sq. in. 100,000[3]	75,000 lb. per sq. in. 125,000[4]
1/16	93	109	202	232
3/32	207	242	449	518
.112	294	343	637	736
1/8	369	431	780	910
.138	447	522	969	1117
5/32	576	672	1248	1440
.164	633	739	1372	1580
3/16	828	966	1794	2000
.190	849	991	1840	2120
7/32	1128	1316	2444	2820
.216	1101	1285	2386	2750
1/4	1473	1719	3192	3570
5/16	2301	2685	4986	5630
3/8	3315	3868	7183	8110
.4375	4509	5261	9770	11,100
1/2	5889	6871	12,760	14,440
5/8	9204	10,738	19,942	23,000
3/4	13,254	15,463	28,717	33,130
7/8	18,039	21.046	39,085	45,100
1	23,562	27,489	51,051	58,900

[1] For aluminum alloy rivets, bolts and pins.

[2] For low carbon steel rivets, bolts and pins.

[3] For bolts, etc., heat-treated to 100,000 lb. per sq. in. tensile strength.

[4] For steel parts heat-treated to 125,000 lb. per sq. in. tensile strength. This includes the parts shown on the following Army-Navy drawings: Hex. head bolts, AN3–AN10; eye bolts, AN42–AN49; clevis pins, AN392-AN398.

TABLE 10 : 2

Bearing Strength of Steel Sheets on Rivets

Allowable Bearing — 95,000 lb. per sq. in.

For use with S. A. E. 1025 Steel Sheet

Size of Rivets	1/16	3/32	1/8	5/32	3/16	1/4	5/16	3/8	1/2	5/8	3/4	7/8	1
Plate Sizes	Bearing Strength of Plate												
.028	166	249	332	415	498	665	831	997	1,330	1,663	1,995	2,328	2,660
.035	208	312	416	520	623	831	1039	1247	1,663	2,078	2,494	2,909	3,325
.049	291	436	582	727	873	1164	1455	1746	2,328	2,909	3,491	4,073	4,655
.058	344	517	689	861	1033	1378	1722	2066	2,755	3,444	4,133	4,821	5,510
.065	386	579	772	965	1158	1544	1929	2316	3,088	3,859	4,631	5,403	6,175
.072	428	641	855	1069	1283	1710	2138	2565	3,420	4,275	5,130	5,985	6,840
.083	493	739	986	1232	1478	1971	2464	2957	3,943	4,928	5,914	6,899	7,885
.095	564	846	1128	1410	1692	2256	2820	3384	4,513	5,641	6,769	7,897	9,025
.120	713	1069	1425	1781	2138	2850	3563	4275	5,700	7,125	8,550	9,975	11,400
3/16	1113	1670	2227	2783	3340	4453	5566	6680	8,906	11,133	13,359	15,586	17,813
1/4	1484	2227	2969	3711	4453	5938	7422	8906	11,875	14,844	17,813	20,781	23,750

TABLE 10 : 3

Bearing Strength of Steel Sheets on Rivets

Allowable Bearing — 100,000 lb. per sq. in.

For use in computing bearing strength at other unit stress values

Size of Rivets	1/16	3/32	1/8	5/32	3/16	1/4	5/16	3/8	1/2	5/8	3/4	7/8	1
Plate Sizes	Bearing Strength of Plate												
.028	175	263	350	438	525	700	875	1050	1,400	1,750	2,100	2,450	2,800
.035	219	328	438	547	656	875	1094	1313	1,750	2,188	2,625	3,063	3,500
.049	306	459	612	766	919	1225	1531	1838	2,448	3,063	3,675	4,288	4,900
.058	362	544	725	906	1087	1450	1812	2175	2,900	3,625	4,350	5,075	5,800
.065	406	609	812	1016	1219	1625	2031	2438	3,250	4,063	4,875	5,688	6,500
.072	450	675	900	1125	1350	1800	2250	2700	3,600	4,500	5,400	6,300	7,200
.083	519	778	1038	1297	1556	2075	2594	3113	4,150	5,188	6,225	7,263	8,300
.095	594	891	1188	1484	1781	2375	2969	3563	4,750	5,938	7,125	8,313	9,500
.120	750	1125	1500	1875	2250	3000	3750	4500	6,000	7,500	9,000	10,500	12,000
3/16	1172	1758	2344	2930	3516	4688	5859	7031	9,375	11,719	14,063	16,406	18,750
1/4	1563	2344	3125	3906	4688	6250	7813	9375	12,500	15,625	18,750	21,875	25,000

TABLE 10 : 4

Bearing Strength of Steel Sheets on Rivets

Allowable Bearing — 140,000 lb. per sq. in.

For use with alloy steel sheet heat-treated to 100,000 lb. per sq. in. tensile strength

Size of Rivets	1/16	3/32	1/8	5/32	3/16	1/4	5/16	3/8	1/2	5/8	3/4	7/8	1
Plate Sizes	Bearing Strength of Plate												
.028	245	368	490	612	735	980	1,225	1,470	1,960	2,450	2,940	3,430	3,920
.035	306	459	612	765	918	1224	1,530	1,836	2,448	3,060	3,672	4,284	4,896
.049	429	643	857	1072	1286	1714	2,143	2,573	3,430	4,288	5,145	6,003	6,860
.058	507	761	1015	1269	1522	2030	2,537	3,045	4,060	5,075	6,090	7,105	8,120
.065	569	853	1137	1422	1706	2275	2,844	3,413	4,550	5,688	6,825	7,963	9,100
.072	630	945	1260	1575	1890	2520	3,150	3,780	5,040	6,300	7,560	8,820	10,080
.083	726	1089	1452	1815	2178	2904	3,630	4,356	5,808	7,260	8,712	10,164	11,616
.095	831	1247	1662	2078	2494	3325	4,156	4,988	6,650	8,313	9,975	11,638	13,300
.120	1050	1575	2100	2625	3150	4200	5,250	6,300	8,400	10,500	12,600	14,700	16,800
3/16	1640	2460	3280	4100	4920	6560	8,200	9,840	13,120	16,400	19,680	22,960	26,240
1/4	2188	3281	4375	5469	6563	8750	10,938	13,125	17,500	21,875	26,250	30,625	35,000

TABLE 10 : 5

Bearing Strength of Aluminum Alloy Sheets on Rivets

Allowable Bearing — 75,000 lb. per sq. in.

Size Rivet or Pin	1/16	3/32	1/8	5/32	3/16	1/4	5/16	3/8	1/2	5/8	3/4	7/3	1″
Plate Thickness	Bearing Strength of Plate												
.014[1]	26	39	52	66	79	105	131	158	210	263	315	368	420
.020[1]	54	80	107	134	161	214	268	321	428	536	643	750	857
.028[1]	105	158	210	263	315	420	525	630	840	1,050	1,260	1,470	1,680
.035	164	246	328	410	492	656	820	984	1312	1,640	1,968	2,296	2,624
.040	188	281	375	469	563	750	938	1125	1500	1,875	2,250	2,625	3,000
.049	230	345	460	575	690	920	1150	1377	1836	2,295	2,754	3,213	3,672
.058	272	408	544	680	816	1088	1360	1631	2174	2,718	3,261	3,805	4,348
.065	305	457	610	762	914	1219	1524	1827	2436	3,045	3,654	4,263	4,872
.083	389	584	778	973	1168	1557	1946	2334	3112	3,890	4,668	5,446	6,224
.095	445	668	891	1114	1336	1782	2227	2670	3560	4,450	5,340	6,230	7,120
.120	563	844	1125	1407	1688	2250	2813	3375	4500	5,625	6,750	7,875	9,000
5/32	732	1098	1464	1830	2196	2928	3660	4395	5860	7,325	8,790	10,255	11,720
3/16	879	1319	1758	2198	2637	3516	4395	5274	7032	8,790	10,548	12,306	14,064
1/4	1172	1758	2344	2930	3516	4688	5860	7035	9380	11,725	14,070	16,415	18,760

[1] In figuring bearing stresses for sheets under 0.035 in.

Let Fb = Allowable bearing on thick plate. t = Thickness of plate. Fb' = Allowable bearing on plate of thickness, t. Then $Fb' = \frac{t}{0.035} Fb$.

The design of riveted joints resolves itself into the determination of the loads on each rivet and the investigation of the rivet to see whether it can transmit these loads in bearing and shear. Bolts and pins are analyzed in the same manner.

10 : 7. Rivets — The normal rivet is of the shape shown in Fig. 10 : 8. One head is formed in the manufacture of the rivet, the other by hammering or pressing after the shank has been pushed through the holes drilled in the sheets to be joined. Bearing between the rivet shank and the sides of the hole together with the shearing strength of the shank prevent relative movement of the members joined in a direction parallel to their plane of contact while movement perpendicular to this plane is resisted by bearing under the sides of the rivet heads and the tensile strength of the shank. When it is undesirable to have the rivet head project beyond the surfaces of the sheets joined, a countersunk rivet such as is shown in Fig. 10 : 10 may be used. If but one head is to be countersunk it is better to have it the one formed in the manufacture of the rivet rather than the one formed in assembling the joint.

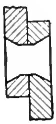


Fig. 10 : 10

When one of the members riveted is a tube, difficulty is encountered in forming the head because of the lack of a solid support for one head while the other is being formed. In some cases this support is obtained by inserting a mandril into the tube to "buck-up" the rivet. In other cases the rivet is made long enough to extend through the tube so that both heads are exposed. Such rivets are liable to be low in strength as the long shank bends under the repeated blows used in forming the head instead of flowing out to fill up the hole in the members. The result is a loose rivet that will not carry much load. When long rivets of this sort must be used it is well to reinforce the shank with a short length of tubing, a "spacer," between the inside walls of the tube. This spacer will prevent mashing of the tube or bending of the rivet while the head is being formed. In many cases the solid rivet may be omitted and the joint formed by "heading" over the ends of a short, heavy-gage tube. Such hollow rivets are generally made of chrome-molybdenum steel tubing and they serve very well in connections between tubing. Sufficient data are not available as to their strength to permit the tabulation of allowable stress data here.

Rivets of both steel and aluminum alloy are used in aircraft work, the material in the rivet generally being the same as that of the members joined, though steel rivets are often used to connect aluminum alloy to steel and sometimes to connect aluminum alloy to aluminum alloy.

10 : 8. Bolting and Pinning — When a rivet head has once been formed it is impossible to remove the rivet without destroying it.

Hence, when joints which must be disassembled are to be made, bolts are used instead of rivets. Bolts are practically always used in wooden members because the ratio of length to diameter is so high that rivets are liable to bend when being headed over, or under the bearing loads which are distributed over the length of the rivet. Bolts may be made of high-strength heat-treated stock so that they can carry higher unit stresses than rivets, [which must be sufficiently soft and ductile that they can be headed.]

Bolts are occasionally made of aluminum alloy but this material is not well adapted for use in screw threads because of its softness and, in addition, due to its low modulus of elasticity and consequent lack of stiffness, aluminum alloy bolts bend easily and have a tendency to concentrate their loads near the points at which they are applied.

Aircraft bolts are generally made with a hexagonal head and provided with a hexagonal nut. The nut is sometimes prevented from coming off by using a lock nut or washer but, due to the vibration of airplane parts, it is much better to safety all important bolts by drilling the shank outside of the nut and inserting a cotter pin. Eye-bolts are often used where it is desired to transmit a load in a direction parallel to the axis of the bolt, as in the case of a wire attachment where the wire has components parallel with and at right angles to the axis of the bolt.

At times it is not desired to use a bolt with a nut but a pin is used that is similar to a bolt without the screw thread. Such pins are safetied by means of cotter pins. In many cases these pins are tapered in diameter and the heads are left off. When tapered pins are used their main purpose is to bring members into line or to obtain a joint which may be tightened from time to time to obtain a perfect bearing between the parts. This is especially true of hinges, and when tapered pins are used the smaller end is generally threaded so that a nut may be applied for drawing up the joint. When tapered pins are used they should always be safetied and if they are employed on main structural members this should be accomplished by a nut and cotter in combination. As an additional safety measure bolts and pins should be inserted with the head or large end uppermost to reduce the possibility of their falling out in the event that they are not properly safetied and the nut drops off. Fittings should always be designed with this in mind.

Bolts, pins and rivets are all analyzed and designed in the same way, proper allowance being made for the different properties of the materials used. In addition to the investigation for strength in bearing and shear it is often necessary to check the strength of long bolts in bending. The threaded portion of a bolt should not be called upon to carry

shear loads if the fitting can be designed to prevent it. When it is necessary for the threaded portion to bear on part of a fitting the load carried on the threads should not exceed 25 per cent of the bolt's shearing strength and when the load so carried is greater than 25 per cent not more than one-fourth of the thickness of the fitting should rest on the threads.

10 : 9. Determination of Loads on Rivets, Bolts or Pins — The methods of determining the loads imposed on each rivet or bolt when several are used in a joint depend on the purpose of the connection and the arrangement of the rivets.

The simplest form of connection to analyze is that where two members carrying simple tension or compression are spliced by inserting one into the other and riveting them together. When such a joint is made, as shown in Fig. 10 : 11, it is assumed that the load is divided equally among the rivets. Having made this assumption it is then a simple matter to determine the load on each rivet and investigate it for shear and bearing. When the members joined carry tension, their cross-sectional areas are reduced by the rivet holes, so it is also necessary to determine the strength of the members themselves after appropriate reduction has been made for the rivet holes. To minimize the reduction in area it is customary to stagger the rivets so that too many will not occur in one cross-section. This need not be done in the case of metal members carrying only compressive loads as the rivets fill their holes and it is possible to transmit compression across them. The net area is, however, generally used on wooden members pierced by rivets, bolts or pins. Moreover, since most members in airplane structures suffer a reversal of loading in the various design conditions it is common practice to stagger rivets, bolts or pins in all members in which they occur.

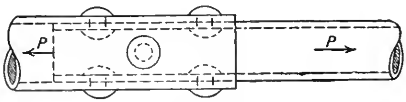
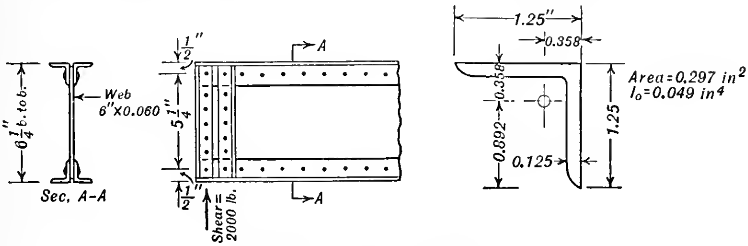


Fig. 10 : 11

In built-up sections which carry bending, such as plate girder spars, the rivets attaching the chord members to the web plate serve in the same capacity as the glued joint between the web and chord of a wooden box spar. Hence, by determining the increment in the longitudinal shearing force between chord and web for a 1-in. strip of spar and by computing the allowable load in shear and bearing on a single rivet, it becomes possible to determine how many rivets must be used per inch to transmit the load from web to chord. This distance between the centers of the rivets, called the pitch, may be determined by the approximate method described under glued joints or by the more exact

method based on $f_s = \frac{SQ}{bI}$. In order to illustrate the analysis of such joints the following problem will be worked out for the aluminum alloy plate girder shown in Fig. 10 : 12.

For the approximate method the distance between the centroids of the chords is $6.25 - 2 \times 0.36 = 5.53$ in., whence $F = \frac{2000 \times 1}{5.53} =$ 362 lb. This represents the longitudinal force between web and chord per inch of span. If we assume 1/8 in. diameter aluminum alloy rivets the limiting bearing would be on the 0.060 web plate. By interpolation in Table 10 : 5 this value is found to be 563 lb. From Table 10 : 1 the allowable shearing strength of a 1/8-in. aluminum alloy rivet is 369 lb.



Aluminum Alloy Plate Girder

Fig. 10 : 12

But these rivets have two sections subjected to shear, one on each side of the web plate, so the total load which the rivet can transmit from web to flange angles is $2 \times 369 = 738$ lb. The limiting load is thus seen to be that carried in bearing, or 563 lb. per rivet. Since each rivet will take 563 lb. and since the force transmitted between web and chord is 362 lb. per in. of span, the rivets can be $\frac{563}{362} = 1.55$ in. between centers.

By the exact method, the force transmitted from web to chord in each inch of span would be:

$$f = \frac{SQ}{I}$$

$$S = 2000 \text{ lb.}$$

$$Q = 2 \times 0.297(3.125 - 0.358) = 1.648 \text{ in.}^3$$

$$I = 4 \times 0.049 + 4 \times 0.297(3.125 - 0.358)^2 + \frac{0.060 \times 6^3}{12}$$

$= 10.38$ in.^{4} if we neglect the small loss due to the rivet holes in the tension chord.

$$f = \frac{2000 \times 1.648}{10.38} = 318 \text{ lb.}$$

From the investigation by the approximate method it was found that the limiting factor was the strength of the rivet bearing on the 0.060 sheet. This was 563 lb. and since the load transmitted between web and chord is 318 lb. per in. the rivets may be spaced $\frac{563}{318} = 1.77$ in. between centers.

Attention is called to the more conservative values obtained in this case by the more approximate and less tedious method.

It is often necessary to reinforce a section such as a girder by adding plates to the top and bottom chords. Such plates, called cover plates, are attached to the flange angles by vertical rivets, and to determine the pitch of these rivets either of the above methods may be applied. If the approximate method is used, the area of the cover plates must, obviously, be considered in the determination of the centroids of the chords. The load to be transmitted between web and chord in each inch of span may then be determined and, assuming that the load carried by the cover plate is directly proportional to the ratio of cover plate area to total chord area, the load to be transmitted between the angles and the cover plate in each inch of span may be readily obtained. Dividing this load by the strength of one rivet then gives the required rivet pitch if there is but one vertical rivet in any section. There are generally two rivets, one through each flange angle, so the pitch will be twice that allowable for one rivet. In using this method it is desirable to use the net area of the tension chord in the computations for rivet pitch in the tension chord although the error involved in the use of the gross area is generally small. If it is desired to use the exact method, the force to be transmitted between the cover plate and the flange angles is obviously equal to the longitudinal shearing force between those members. Hence, by determining this force, $\frac{SQ}{I}$, and dividing by the strength of the one or two rivets used to attach the cover plates at the section being investigated the required pitch is readily obtained.

Since the pitch of the rivets attaching the chords to the webs or the cover plates to the flange angles is a function of the shear at the section being considered it is obvious that the required pitch will vary at different sections. It is therefore necessary to determine the pitch at all sections where the shear is a maximum, so that sufficient rivets may be provided, and at enough intermediate sections so that advantage may be taken of any permissible increases in pitch. In order to avoid the possibility of local buckling of the individual members of the beam, it is recommended that normal structural design practice be followed

regarding the maximum pitch used. This would limit that pitch to about 8 times the diameter of the rivet, or 16 times the thickness of the thinnest plate connected, whichever is the smaller.

10 : 10. Riveted Joints Carrying Torsion or Bending — In many cases, notably the fittings attaching wing beams to the fuselage structure, it is necessary to transmit torsion or bending as well as direct stress by means of rivets or bolts. Let us consider the stresses in the rivets of such a fitting for a duralumin girder spar carrying the loads shown in Fig. 10 : 13.

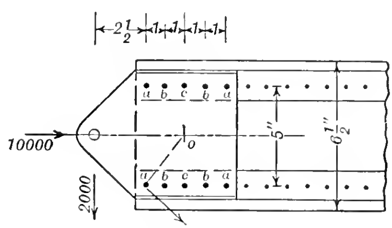


FIG. 10 : 13

Each of the rivets in this fitting carries a part of the horizontal load of 10,000 lb., part of the vertical load of 2000 lb., which represents the shear at the end of the spar, and part of the moment developed by the shear which is applied at some distance, 4.5 in. in this case, from the center of resistance of the group of rivets. When a joint of this type is made between isotropic materials this center of resistance is at the center of gravity of the group.

Assuming that each of the ten rivets carries an equal part of the horizontal and vertical loads, it follows that there is a horizontal component of 1000 lb. and a vertical component of 200 lb. on each rivet. This assumption is reasonable and is well substantiated by test data.

Assuming that the force produced on each rivet by the moment caused by the eccentric load of 2000 lb. varies as the distance of that rivet from the center of resistance of the group and that this force acts at right angles to a line between the rivet and the center of resistance, it follows that the resistance to torsion afforded by that rivet, such as one of the a rivets, would be, $R = f(oa)$ where f represents the force on the rivet and oa is its distance from the center of the group. For rivet b, $R = f\left(\frac{bo}{oa}\right)bo$, and for rivet c, $R = f\left(\frac{co}{oa}\right)co$. Then the resistance afforded by the group of rivets shown is,

$$R = \frac{f}{oa}(4\,\overline{oa}^2 + 4\,\overline{ob}^2 + 2\,\overline{oc}^2),$$

which, when rearranged, may be written,

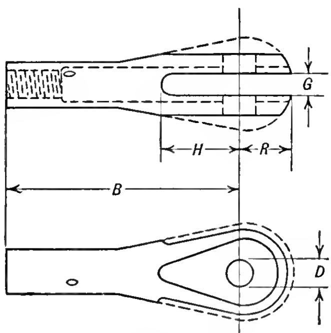
$$f = \frac{R(oa)}{4\,\overline{oa}^2 + 4\,\overline{ob}^2 + 2\,\overline{oc}^2} = \frac{Md}{I}$$

where f is the load on the most stressed rivet, M is the torsional moment on the group of rivets measured about the center of resistance of the

group, d is the distance from the center of resistance to the most stressed rivet and, I is the summation of the squares of the distances from the center of resistance of the group to each rivet.

In the case under consideration, $M = 2000(2.0 + 2.5) = 9000$ in.-lb., $d = \sqrt{2.0^2 + 2.5^2} = 3.20$ in., $I = 4(\sqrt{2.0^2 + 2.5^2})^2 + 4(\sqrt{1.0^2 + 2.5^2})^2 + 2(2.5)^2 = 82.5$ from which, $f = \dfrac{9000 \times 3.20}{82.5} = 349$ lb. on each of the a rivets. On the lower left a rivet this load will act as shown by the arrow, so that both its horizontal and vertical components act in the same direction as those due to the 10,000-lb. and 2000-lb. loads. Hence this rivet is the most stressed of the group. It is subjected to a total horizontal force of $1000 + \dfrac{2.5}{3.2}\,349 = 1272$ lb. and a total vertical force of $200 + \dfrac{2.0}{3.2}\,349 = 418$ lb. The resultant force on the rivet is then $\sqrt{1272^2 + 418^2} = 1340$ lb. With the force acting on the most stressed rivet known it is possible to determine whether or not the group has sufficient strength by investigating that rivet for bearing and shear in the usual manner.

As was mentioned above, when the joint is made in isotropic materials the center of resistance of the rivet group is at the center of gravity of the group and the assumption that the load carried by each rivet varies as the distance from the center of resistance is valid. But when a joint carrying bending is made in a material such as wood where the stress-strain ratio varies according to the angle which the load makes to the grain, this assumption is no longer valid. In such a case the load distribution is really indeterminate but it is customary to investigate the joint by the foregoing method and to allow ample margin of safety to provide for the differences in load distribution. A more satisfactory procedure is to build up a joint and test it to destruction.



Standard Clevis

FIG. 10 : 14

10 : 11. Other Types of Mechanical Joints — Swaged tie-rods and streamline wires are threaded at each end for the attachment of the standard clevis ends used to connect the wire to a lug or other fitting. Such threaded ends are standard for the various sizes of wire and require no investigation by the airplane designer. Fig. 10 : 14 shows a standard clevis and Table 10 : 6 gives the general dimensions and allowable loads on the standard sizes for swaged and streamline wire.

TABLE 10 : 6

DIMENSIONS OF STANDARD CLEVISES

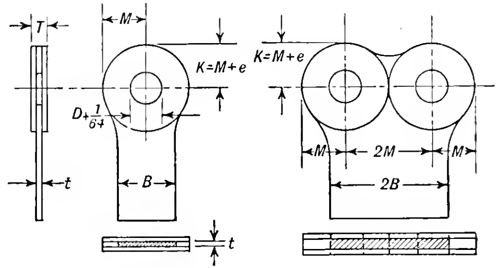
Size	Rated Strength, lb.	*B* in.	*D* in.	*G* in.	*H* in.	*R* in.
6–40	1,000	1 5/16	0.188	0.109	3/8	7/32
10–32	2,100	1 17/32	0.188	0.150	15/32	9/32
1/4–28	3,400	1 13/16	0.250	0.203	5/8	23/64
5/16–24	4,600	1 7/8	0.313	0.203	21/32	25/64
5/16–24	6,100	2	0.375	0.203	27/32	7/16
3/8–24	8,000	2 1/4	0.375	0.266	7/8	1/2
7/16–20	11,500	2 1/2	0.438	0.344	1	39/64
1/2–20	15,500	2 13/16	0.500	0.406	1 3/16	43/64
9/16–18	20,200	3 1/8	0.563	0.453	1 3/8	25/32
5/8–18	24,700	3 3/8	0.625	0.516	1 1/2	27/32

From Army-Navy Standard Sheet 665.

Table 10 : 7 gives the dimensions of lugs for use with the standard sizes of wire. These lugs will take the standard clevis ends and, if made of cold-rolled steel (S. A. E. specification 1025), will develop the full rated strength given in the table. The substitution of alloy steel for cold-rolled is not recommended if it is to be accompanied by a reduction in the dimensions of these lugs. The saving in weight is small and is offset by the element of danger introduced by the possible substitution of the lower strength material for the alloy steel in making a field repair.

TABLE 10 : 7

LIMITING DIMENSIONS FOR LUGS, S. A. E. 1025 — COLD ROLLED STEEL[1]



No.	Rated Strength	D	K	M	e	T	Width of Shank B: t=1/32	1/20	1/16	3/32	1/8	3/16	1/4	5/16	3/8
11	1,100	3/16	17/64	7/32	3/64	3/32	3/4	1/2	3/8	1/4					
21	2,100	3/16	11/32	9/32	1/16	1/8	1 7/16	15/16	3/4	1/2	3/8				
34	3,400	1/4	13/32	11/32	1/16	3/16	2 5/16	1 7/16	1 3/16	13/16	5/8	7/16			
46	4,600	5/16	17/32	7/16	3/32	3/16		1 15/16	1 9/16	1 1/16	13/16	9/16			
61	6,100	3/8	45/64	9/16	9/64	3/16		2 9/16	2 1/16	1 3/8	1 1/16	3/4			
80	8,000	3/8	45/64	9/16	9/64	1/4			2 7/8	1 15/16	1 7/16	1	3/4		
125	12,500	7/16	13/16	21/32	5/32	5/16				2 13/16	2 1/8	1 7/16	1 1/16	7/8	
175	17,500	1/2	15/16	3/4	3/16	3/8					2 3/8	1 3/4	1 5/16	1 1/16	7/8

[1] The values in this table may be revised in the near future in order to adopt identical standards for the Army and Navy.

From U. S. Army Air Corps "Handbook of Instructions for Airplane Designers."

10 : 12. Clamping — Clamping forms one of the more important of the secondary types of connections used in airplanes. As it is considered poor practice to weld any member, particularly a tube, subjected to bending or compression except at the points of support, many items of equipment are attached to the main members of the fuselage by clamps. Such clamps are generally made to encircle the member to which they are attached so that they will not produce local deformations or cause failures in the member.

Clamps depend for their strength on the friction between them and the member to which they are attached and as that depends in turn on the materials used and on the tightness of the clamp an exact analysis of such joints cannot be made. For this reason clamps are never used for connections between main structural members. When used to attach equipment or secondary structure to main members a careful analysis must be made of the secondary stresses produced in those members by the concentrated loads thus imposed on them.

10 : 13. Sewing — Fabric covering is attached to the wings of an airplane by sewing. The reliability and strength of such a method of attachment depend to a great extent on the strength of the thread and of the knots which are made as part of the stitch used. This reliability and strength are considerably augmented by the adhesive action of the dope between thread and fabric, so it is difficult to classify sewing as belonging to any one type of connection. Except for the attachment of the covering to the wings, fuselage and tail surfaces, the lacing or sewing of members is not practiced in airplane construction.

10 : 14 Lashing — Lashing is seldom used on an airplane except for the temporary attachment of light equipment to members in the fuselage. The nearest approach to it that is used on main structural members is to be found in the terminals used for hard wire, often called "piano" wire. In such terminals the end of the wire is bent back to form a loop and is then lashed by being wrapped with wire to form a ferrule. The whole joint is then tinned to prevent motion of the parts; such a joint depends for its strength partly on the friction of the ferrule, partly on the adhesive action of the tin or solder used to keep the end in place. Such terminals vary greatly in strength and they should never be assumed to develop more than 85 per cent of the strength of the wire. In forming the loop care must be taken that the radius of bend is not too small or the wire, which is hard and brittle, will crack and break under a very low load.

10 : 15. Splicing — Splicing is the form of joint used on cables. The end of the cable is bent back over a thimble to form a loop or "eye" and the strands at the end are unraveled for a short distance

and interlaced with those of the standing part in such a way that the resulting joint may develop the full strength of the cable, but its average efficiency is only about 85 per cent. It is customary in aircraft construction to tin a splice after it is made as an added precaution against slipping of the strands.

10 : 16. Nails, etc. — Nails, wood screws, and machine screws act in the same way as pins or bolts which are prevented from falling out by the friction between them and the material in which they are embedded. Neither nails nor screws are sufficiently reliable to be counted upon to carry load, especially if vibration is present. When used in connection with glue or some other type of joint nails or screws should not be assumed to carry any load and, so far as is possible, the use of either should be avoided in the joints used on the primary members of an airplane structure. When they are used in wood the effective areas of the wood members should be reduced by the projected areas of the nails or screws used at any section.

PROBLEMS

10 : 1. Assuming the tubes shown in Fig. 10 : 11 to be of chrome-molybdenum steel and the sizes to be 1 1/8–0.065 and 1 1/4–0.058, what is the maximum allowable value for load P if there are six 1/8 in. diameter steel rivets as indicated?

10 : 2. Determine the sizes of clevis ends for the streamline wires designed in Problem **9 : 2** and design lugs of cold rolled steel suitable for the attachment of these wires to the fuselage structure. Follow the standard practice of making the lugs sufficiently strong to carry the full rated strength of the wires used.

10 : 3. Design fittings for all wires in the drag truss for which the stresses are shown in Fig. 7 : 14. Assume them to be cut from cold rolled steel sheet.

10 : 4. Using the type of end fitting shown in Fig. 10 : 5 and assuming the center of the pin hole to be 2 1/2 in. from the end of the spar design a fitting to carry an axial load of 10,000 lb. and a downward shearing force of 1000 lb. for a beam having the dimensions shown in Fig. 10 : 7. Assume plates of cold rolled steel 1/8 in. thick, reinforced by washers welded on at the pin hole if necessary, the filler block to be of spruce with grain parallel to the spar axis and the bolts to be standard aircraft bolts of nickel steel heat-treated to 100,000 lb. per sq. in.

CHAPTER XI

COMBINED BENDING AND COMPRESSION

The most interesting type of airplane member that the structural engineer is called upon to design is that subjected simultaneously to axial and bending loads, particularly those in which the axial load is one of compression. Such members are very common in all types of structural work, but in most fields it is allowable to make conservative assumptions which permit the members to be designed with little more difficulty than is involved in the investigation of an ordinary column or simple beam. As such assumptions must be conservative, they involve making the members heavier than necessary, and since the importance of economy of weight is greater in airplane design than in most structural work the aeronautical engineer is not satisfied with their use. Furthermore, the airplane designer is in the habit of using long and slender columns for which the usual assumptions that are satisfactory in bridge and building design are not conservative, but exceedingly unsafe.

Basic formulas for beams subjected to combined loads were published by Müller-Breslau in 1902 or earlier, but they were so difficult to use and of so little practical advantage in structural work that they were neglected. During the War, Müller-Breslau worked out his formulas in more detail and supplied tables of complex functions that made possible their use in practical design. At the same time the English, working independently, developed similar equations and tables of functions. The English equations and tables were put into their most useful form by Arthur Berry and are usually known by his name.

The use of the Berry formulas was adopted by the United States Navy during the War, but the Army continued to use certain approximate formulas. In 1922 a thorough study of the problem was begun by the Engineering Division of the Army Air Corps in which it was found that the Berry and Müller-Breslau formulas were fundamentally identical, the only differences being in nomenclature and in the location of the origin. Neither set of formulas was adopted, however, as it was believed that a set resting on the same theoretical basis but much easier to apply could be developed. Such formulas were worked out by the present authors for the case covered by Berry and the more important ones covered by Müller-Breslau, and also for certain cases

often arising in airplane design but covered by neither. At the same time sufficient experimental work was done to demonstrate that the use of the formulas was justified.

11 : 1. General Effect of an Axial Load — Due to the side load alone, the strut, Fig. 11 : 1, would deflect a distance, y', at a distance, x, from the left support, and the bending moment at that point would be,

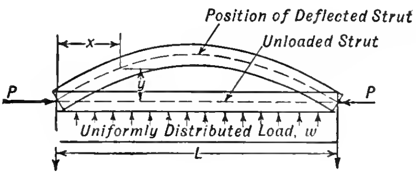


Fig. 11 : 1

$$M = \frac{wx^2}{2} - \frac{wLx}{2}$$

If a compressive load, P, were to be applied as shown, the moment at x would be increased by $-Py'$, since the load, P, would act at a distance, y', from the axis of the deflected member. This increase in moment would cause a greater deflection at x which, in turn, would result in a further increase in the moment. If the load, P, were not too great, these increments of the moment and deflection would become smaller and smaller until eventually, the strut would reach a state of equilibrium. If, however, the load, P, were sufficiently large, the increments of deflection would be successively greater and greater until failure occured. It is apparent, then, that it should be possible to represent increments of deflection or bending moment by a mathematical series of some kind which, if the axial compression were not too great, would converge, so that the limit of the series could be taken to represent conditions when the member reached a state of equilibrium.

The bending moment that would be caused by the lateral load acting alone is called the "primary bending moment," and the deflection under that load acting alone is the "primary deflection." The sums of the infinite series of increments of bending moment and deflection due to the interaction of the axial and lateral loads are called the "secondary" bending moments and deflections.

If P were tension instead of compression, the deflection, y', due to the side load alone would be reduced instead of increased; and, as P was increased, the beam would tend to straighten and the moment at any point would be reduced. The failure, when it occurred, would be a tension failure, and there would be no tendency toward elastic instability or buckling as would be the case with a compressive axial load.

Thus it is evident that an axial compressive load, which increases the bending moment at every point, is of far greater importance in the design of members under combined loads than an axial tension, which

tends to decrease the bending moment. For this reason the formulas developed in this chapter have been confined almost exclusively to cases including a compressive load, although some attention has been given to axial tension.

If the member is continuous over two or more supports, the bending moments will be increased throughout by the application of an axial compressive load. The ordinary three-moment equation, which would be used on a continuous beam in conjunction with any of the standard approximate methods for evaluating the effect of the axial load, makes no provision for this change in moment over the supports and so vitiates the effect of any power series or other device used in such formulas. The methods developed in this chapter, however, provide for the axial load both in the three-moment equation and in the formulas for the moments in the spans by the use of mathematical series. It so happens that the series used with axial compression are identical with those of the trigonometric functions, sines, cosines, and tangents.

In working with the formulas of this chapter the term "sin x" should always be thought of as representing the infinite series

$$\sin x = x - \frac{x^3}{!3} + \frac{x^5}{!5} - \frac{x^7}{!7} + \cdots$$

and "cos x" the series

$$\cos x = 1 - \frac{x^2}{!2} + \frac{x^4}{!4} - \frac{x^6}{!6} + \cdots$$

instead of ratios between the sides of a right triangle. The other functions such as tan x and sec x should be thought of as quotients, reciprocals, etc., of the basic sin x and cos x series, the relationships being those with which the reader is familiar from his study of trigonometry. It is quite correct mathematically to consider the relationships between the sides of a right triangle usually employed to define the sine and cosine of x to be coincidences that are true when one angle of a right triangle is x radians in magnitude. If the reader will get the habit of thinking of the sine and other "trigonometrical" functions as abbreviations for the corresponding infinite series, he will find it much easier to see the justification for the use of the formulas and will not spend his time looking in vain for angles that do not exist, nor will he doubt the validity of the formulas because the angles cannot be found. If sin x is thought of as a trigonometrical relationship only, its appearance in the formulas will seem absurd, while if thought of as an infinite series its appearance becomes most appropriate.

11 : 2. Derivation of the Formulas — The derivation of the formulas for the most common type of loading, a uniformly distributed lateral load in conjunction with an axial compression, is carried out in detail in this and the next article while the expressions for other types of loading are given in Art. 11 : 5, so they may be used when needed in design work. For brevity, computations involving only simple algebra or arithmetic are omitted.

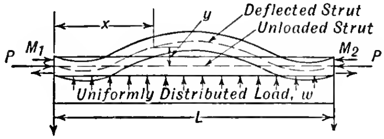


FIG. 11 : 2

Fig. 11 : 2 shows a member supported at two points and subjected to a uniformly distributed transverse load, an axial compression, and restraining moments applied at the points of support.

The expression for the moment at any point is

$$M = M_1 + \frac{(M_2 - M_1)}{L}x - \frac{wLx}{2} + \frac{wx^2}{2} - Py \qquad 11:1$$

By making the usual assumptions of the beam theory

$$M = \frac{EI\,d^2y}{dx^2}$$

whence

$$EI\frac{d^2y}{dx^2} + Py = M_1 + \frac{M_2 - M_1}{L}x - \frac{wLx}{2} + \frac{wx^2}{2}$$

If differentiated twice with respect to x, this becomes

$$\frac{d^2M}{dx^2} + \frac{P}{EI}M = w$$

or, writing $\frac{1}{j^2}$ for $\frac{P}{EI}$, j being $\sqrt{\frac{EI}{P}}$,

$$\frac{d^2M}{dx^2} + \frac{1}{j^2}M = w$$

The solution for this differential equation is

$$M = C_1 \sin\frac{x}{j} + C_2 \cos\frac{x}{j} + wj^2 \qquad 11:2$$

C_1 and C_2 being the constants of integration.

When $x = 0$, $M = M_1$, and when $x = L$, $M = M_2$; hence,

$$C_1 = \frac{M_2 - wj^2}{\sin\frac{L}{j}} - \frac{M_1 - wj^2}{\tan\frac{L}{j}} = \frac{M_2 - wj^2 - (M_1 - wj^2)\cos\frac{L}{j}}{\sin\frac{L}{j}}$$

and

$$C_2 = M_1 - wj^2$$

For brevity:

$$D_1 = M_1 - wj^2$$

and

$$D_2 = M_2 - wj^2$$

The moment at any point on the span is

$$M = \frac{D_2 - D_1 \cos \frac{L}{j}}{\sin \frac{L}{j}} \sin \frac{x}{j} + D_1 \cos \frac{x}{j} + wj^2 \qquad 11:3$$

To find the location of the section of maximum moment, differentiate equation 11 : 2, equate the first derivative to zero, and solve; whence,

$$\frac{dM}{dx} = 0 = \frac{C_1}{j} \cos \frac{x}{j} - \frac{C_2}{j} \sin \frac{x}{j}$$

and

$$\tan \frac{x}{j} = \frac{C_1}{C_2} = \frac{D_2 - D_1 \cos \frac{L}{j}}{D_1 \sin \frac{L}{j}} \qquad 11:4$$

From the value of $\frac{x}{j}$, determined from equation 11 : 4 and the use of Table 11 : 1, the distance, x, to the section of maximum moment, is readily obtained. The value of x, obtained in this way, must lie between zero and L. Otherwise, either M_1 or M_2 is the maximum on the member.

The maximum moment may be found by substituting the value from equation 11 : 4 in equation 11 : 3 and simplifying:

$$M_{\max} = \frac{D_1}{\cos \frac{x}{j}} + wj^2 \qquad 11:5$$

The bending moment determined from equation 11 : 5 is the moment at the section where the slope of the moment curve changes sign. It may be either a maximum or a minimum numerically, and may also be smaller than the bending moments at the ends of the span. Even though it is not the largest bending moment on the beam the margin of safety at its section should always be investigated as it is likely to be critical, the allowable unit stress being smaller between supports that at a support.

The deflection at any point is found by substituting the value of M from equation 11 : 2 in equation 11 : 1 and solving for y,

$$y = \frac{1}{P}\left(M_1 + \frac{M_2 - M_1}{L}x - \frac{wLx}{2} + \frac{wx^2}{2} - \frac{D_2 - D_1 \cos\frac{L}{j}}{\sin\frac{L}{j}} \sin\frac{x}{j} - D_1 \cos\frac{x}{j} - wj^2\right) \qquad 11:6$$

The first derivative of equation 11 : 6 gives the slope of the tangent to the elastic curve at any point:

$$i = \frac{1}{P}\left(\frac{M_2 - M_1}{L} - \frac{wL}{2} + wx - \frac{C_1}{j}\cos\frac{x}{j} + \frac{C_2}{j}\sin\frac{x}{j}\right) \qquad 11:7$$

11 : 3. Equation of Three Moments — In two contiguous spans, shown in Fig. 11 : 3, the slope of the tangent to the elastic curve at the center support will be the same for both spans, the member being continuous over this support.

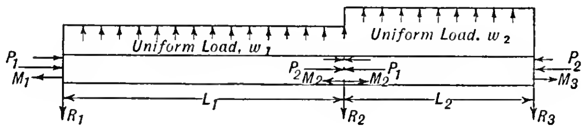


FIG. 11 : 3

At R_2, $x_1 = L$, for the left-hand, and $x_2 = 0$ for the right-hand span. Using subscripts or primes to differentiate between the symbols for the respective spans and substituting these values in the expressions for slope at R_2,

$$i_1 = \frac{M_2 - M_1}{L_1P_1} - \frac{w_1L_1}{2\,P_1} + \frac{w_1L_1}{P_1} - \frac{C_1\cos\frac{L_1}{j_1}}{j_1P_1} + \frac{C_2\sin\frac{L_1}{j_1}}{j_1P_1} \qquad 11:8$$

in which

$$C_1 = \frac{M_2 - w_1j_1^2 - (M_1 - w_1j_1^2)\cos\frac{L_1}{j_1}}{\sin\frac{L_1}{j_1}}$$

and

$$C_2 = M_1 - w_1j_1^2$$

$$i_2 = \frac{M_3 - M_2}{L_2P_2} - \frac{w_2L_2}{2\,P_2} - \frac{C_1'}{j_2P_2} \qquad 11:9$$

in which

$$C'_1 = \frac{M_3 - w_2 j_2^2 - (M_2 - w_2 j_2^2)\cos\frac{L_2}{j_2}}{\sin\frac{L_2}{j_2}}$$

However, $i_1 = i_2$ at the center support. Substituting the values for C_1, C'_1 and C_2 in equations 11 : 8 and 11 : 9, combining the terms, and simplifying, the following result is obtained:

$$\frac{M_1 L_1}{I_1}\left(\frac{\frac{L_1}{j_1}\operatorname{cosec}\frac{L_1}{j_1} - 1}{\left(\frac{L_1}{j_1}\right)^2}\right) + \frac{M_3 L_2}{I_2}\left(\frac{\frac{L_2}{j_2}\operatorname{cosec}\frac{L_2}{j_2} - 1}{\left(\frac{L_2}{j_2}\right)^2}\right)$$

$$+M_2\left\{\frac{L_1}{I_1}\left(\frac{\left(1 - \frac{L_1}{j_1}\cot\frac{L_1}{j_1}\right)}{\left(\frac{L_1}{j_1}\right)^2}\right) + \frac{L_2}{I_2}\left(\frac{\left(1 - \frac{L_2}{j_2}\cot\frac{L_2}{j_2}\right)}{\left(\frac{L_2}{j_2}\right)^2}\right)\right\}$$

$$= \frac{w_1 L_1^3}{I_1}\left(\frac{\tan\frac{L_1}{2\,j_1} - \frac{L_1}{2\,j_1}}{\left(\frac{L_1}{j_1}\right)^3}\right) + \frac{w_2 L_2^3}{I_2}\left(\frac{\tan\frac{L_2}{2\,j_2} - \frac{L_2}{2\,j_2}}{\left(\frac{L_2}{j_2}\right)^3}\right) \qquad 11:10$$

Multiplying equation 11 : 10 by 6, it becomes

$$\frac{M_1 L_1 \alpha_1}{I_1} + 2\,M_2\left(\frac{L_1}{I_1}\beta_1 + \frac{L_2}{I_2}\beta_2\right) + \frac{M_3 L_2 \alpha_2}{I_2}$$
$$= \frac{w_1 L_1^3}{4\,I_1}\gamma_1 + \frac{w_2 L_2^3}{4\,I_2}\gamma_2 \qquad 11:11$$

in which, with subscripts for L and j corresponding to the spans:

$$\alpha = 6\,\frac{\left(\frac{L}{j}\operatorname{cosec}\frac{L}{j} - 1\right)}{\left(\frac{L}{j}\right)^2}$$

$$\beta = 3\,\frac{\left(1 - \frac{L}{j}\cot\frac{L}{j}\right)}{\left(\frac{L}{j}\right)^2}$$

$$\gamma = 3\,\frac{\left(\tan\frac{L}{2\,j} - \frac{L}{2\,j}\right)}{\left(\frac{L}{2\,j}\right)^3}$$

The sines, cosines, and tangents of $\frac{L}{j}$, and also the values of α, β, and γ, corresponding to the different values of $\frac{L}{j}$, will be found in Tables 11 : 1 and 11 : 3.

In Chapter IV, it was shown how the Three-Moment Equation for beams subjected to no axial load could be written for the cases in which an external moment is applied at the intermediate support. Formula 11 : 11 can be similarly modified to apply to the corresponding cases when axial compression is present, giving

$$\frac{M_1 L_1 \alpha_1}{I_1} + 2\,M_{-2}\frac{L_1}{I_1}\beta_1 + 2\,M_{+2}\frac{L_2}{I_2}\beta_2 + \frac{M_3 L_2 \alpha_2}{I_2} = \frac{w_1 L_1^3 \gamma_1}{4\,I_1} + \frac{w_2 L_2^3 \gamma_2}{4\,I_2} \qquad 11:12$$

In Formula 11 : 12, M_{-2} and M_{+2} are the moments an infinitesimal distance to the left and right, respectively, of the point of support. Formula 11 : 12 contains an additional unknown which necessitates another equation for a solution. This is obtained from the relation between M_{-2} and M_{+2}, M_{+2} being equal to M_{-2} plus or minus the eccentric moment, M_e. Care must be taken with the sign of M_e. It should be considered positive if it increases the moment from M_{-2} to M_{+2} as one goes from left to right over the point of support.

In Formula 4 : 1 (see Chapter IV) two terms are shown on the right-hand side of the equation to allow for the effect of deflection of the supports. The same two terms may be properly added to the right-hand side of Formula 11 : 11 for the same purpose. No modification of these deflection terms is needed when an axial load is present.

TABLE 11 : 1

NATURAL SINES, COSINES, AND TANGENTS OF ANGLES IN RADIANS

L/j in radians	Sine $\frac{L}{j}$	Cosine $\frac{L}{j}$	Tangent $\frac{L}{j}$	L/j in radians	Sine $\frac{L}{j}$	Cosine $\frac{L}{j}$	Tangent $\frac{L}{j}$
0.00	0.00000	1.00000	0.00000	0.50	0.47943	0.87758	0.54630
0.01	0.01000	0.99995	0.01000	0.51	0.48818	0.87274	0.55936
0.02	0.02000	0.99980	0.02000	0.52	0.49688	0.86782	0.57256
0.03	0.03000	0.99955	0.03000	0.53	0.50553	0.86281	0.58591
0.04	0.03999	0.99920	0.04002	0.54	0.51414	0.85771	0.59943
0.05	0.04998	0.99875	0.05004	0.55	0.52269	0.85252	0.61310
0.06	0.05996	0.99820	0.06007	0.56	0.53119	0.84726	0.62695
0.07	0.06994	0.99755	0.07012	0.57	0.53963	0.84190	0.64097
0.08	0.07991	0.99680	0.08017	0.58	0.54802	0.83646	0.65517
0.09	0.08988	0.99595	0.09024	0.59	0.55636	0.83094	0.66955
0.10	0.09983	0.99500	0.10034	0.60	0.56464	0.82534	0.68414
0.11	0.10978	0.99396	0.11045	0.61	0.57287	0.81965	0.69892
0.12	0.11971	0.99281	0.12057	0.62	0.58104	0.81388	0.71391
0.13	0.12963	0.99156	0.13073	0.63	0.58914	0.80803	0.72911
0.14	0.13954	0.99022	0.14092	0.64	0.59720	0.80210	0.74454
0.15	0.14944	0.98877	0.15114	0.65	0.60519	0.79608	0.76021
0.16	0.15932	0.98723	0.16138	0.66	0.61312	0.78999	0.77611
0.17	0.16918	0.98558	0.17165	0.67	0.62099	0.78382	0.79226
0.18	0.17903	0.98384	0.18197	0.68	0.62879	0.77757	0.80866
0.19	0.18886	0.98200	0.19232	0.69	0.63654	0.77125	0.82533
0.20	0.19867	0.98007	0.20271	0.70	0.64422	0.76484	0.84229
0.21	0.20846	0.97803	0.21314	0.71	0.65183	0.75836	0.85953
0.22	0.21823	0.97590	0.22362	0.72	0.65938	0.75181	0.87707
0.23	0.22798	0.97367	0.23414	0.73	0.66687	0.74517	0.89492
0.24	0.23770	0.97134	0.24472	0.74	0.67429	0.73847	0.91309
0.25	0.24740	0.96891	0.25534	0.75	0.68164	0.73169	0.93160
0.26	0.25708	0.96639	0.26602	0.76	0.68892	0.72484	0.95045
0.27	0.26673	0.96377	0.27676	0.77	0.69614	0.71791	0.96967
0.28	0.27636	0.96106	0.28756	0.78	0.70328	0.71091	0.98926
0.29	0.28595	0.95824	0.29841	0.79	0.71035	0.70385	1.00924
0.30	0.29552	0.95534	0.30934	0.80	0.71736	0.69671	1.02964
0.31	0.30506	0.95233	0.32032	0.81	0.72429	0.68950	1.05046
0.32	0.31457	0.94924	0.33139	0.82	0.73115	0.68222	1.07171
0.33	0.32404	0.94604	0.34253	0.83	0.73793	0.67488	1.09343
0.34	0.33349	0.94275	0.35374	0.84	0.74464	0.66746	1.11563
0.35	0.34290	0.93937	0.36503	0.85	0.75128	0.65998	1.13834
0.36	0.35227	0.93590	0.37640	0.86	0.75784	0.65244	1.16155
0.37	0.36162	0.93233	0.38786	0.87	0.76433	0.64483	1.18533
0.38	0.37092	0.92866	0.39941	0.88	0.77074	0.63715	1.20967
0.39	0.38019	0.92491	0.41105	0.89	0.77707	0.62941	1.23460
0.40	0.38942	0.92106	0.42279	0.90	0.78333	0.62161	1.26016
0.41	0.39861	0.91712	0.43463	0.91	0.78950	0.61375	1.28637
0.42	0.40776	0.91309	0.44657	0.92	0.79560	0.60582	1.31326
0.43	0.41687	0.90897	0.45862	0.93	0.80162	0.59783	1.34088
0.44	0.42594	0.90475	0.47078	0.94	0.80756	0.58979	1.36923
0.45	0.43497	0.90045	0.48306	0.95	0.81342	0.58168	1.39838
0.46	0.44395	0.89605	0.49545	0.96	0.81919	0.57352	1.42836
0.47	0.45289	0.89157	0.50796	0.97	0.82489	0.56530	1.45920
0.48	0.46178	0.88699	0.52061	0.98	0.83050	0.55702	1.49096
0.49	0.47063	0.88233	0.53339	0.99	0.83603	0.54869	1.52368
0.50	0.47943	0.87758	0.54630	1.00	0.84147	0.54030	1.55741

TABLE 11 : 1 — (*Continued*)

NATURAL SINES, COSINES, AND TANGENTS OF ANGLES IN RADIANS

L/j in radians	Sine $\frac{L}{j}$	Cosine $\frac{L}{j}$	Tangent $\frac{L}{j}$	L/j in radians	Sine $\frac{L}{j}$	Cosine $\frac{L}{j}$	Tangent $\frac{L}{j}$
1.00	0.84147	0.54030	1.55741	1.50	0.99749	0.07074	14.10142
1.01	0.84683	0.53186	1.59221	1.51	0.99815	0.06076	16.42811
1.02	0.85211	0.52337	1.62813	1.52	0.99871	0.05077	19.6696
1.03	0.85730	0.51482	1.66525	1.53	0.99917	0.04079	24.4986
1.04	0.86240	0.50622	1.70361	1.54	0.99953	0.03079	32.4513
1.05	0.86742	0.49757	1.74332	1.55	0.99978	0.02079	48.0803
1.06	0.87236	0.48887	1.78442	1.56	0.99994	0.01080	92.6238
1.07	0.87720	0.48012	1.82703	1.57	1.00000	0.00080	1,275.04
1.08	0.88196	0.47133	1.87122	1.58	0.99996	−0.00920	−108.661
1.09	0.88663	0.46249	1.91710	1.59	0.99982	−0.01920	−52.0676
1.10	0.89121	0.45360	1.96476	1.60	0.99957	−0.02920	−34.2329
1.11	0.89570	0.44466	2.01434	1.61	0.99923	−0.03919	−25.4950
1.12	0.90010	0.43568	2.06595	1.62	0.99879	−0.04918	−20.3073
1.13	0.90441	0.42666	2.11975	1.63	0.99825	−0.05917	−16.8712
1.14	0.90863	0.41759	2.17588	1.64	0.99760	−0.06915	−14.4270
1.15	0.91276	0.40849	2.23449	1.65	0.99687	−0.07912	−12.59926
1.16	0.91680	0.39934	2.29580	1.66	0.99602	−0.08909	−11.18059
1.17	0.92075	0.39015	2.35998	1.67	0.99508	−0.09904	−10.04724
1.18	0.92461	0.38092	2.42726	1.68	0.99404	−0.10899	−9.12076
1.19	0.92837	0.37166	2.49790	1.69	0.99290	−0.11892	−8.34925
1.20	0.93204	0.36236	2.57215	1.70	0.99167	−0.12884	−7.69660
1.21	0.93562	0.35302	2.65033	1.71	0.99033	−0.13875	−7.13723
1.22	0.93910	0.34365	2.73276	1.72	0.98889	−0.14865	−6.65245
1.23	0.94249	0.33424	2.81982	1.73	0.98736	−0.15853	−6.22809
1.24	0.94578	0.32480	2.91194	1.74	0.98572	−0.16840	−5.85353
1.25	0.94898	0.31532	3.00957	1.75	0.98399	−0.17825	−5.52037
1.26	0.95209	0.30582	3.11328	1.76	0.98215	−0.18808	−5.22209
1.27	0.95510	0.29628	3.22363	1.77	0.98023	−0.19789	−4.95340
1.28	0.95802	0.28672	3.34135	1.78	0.97819	−0.20768	−4.71010
1.29	0.96084	0.27712	3.46721	1.79	0.97607	−0.21745	−4.48866
1.30	0.96356	0.26750	3.60210	1.80	0.97385	−0.22720	−4.28627
1.31	0.96618	0.25785	3.74708	1.81	0.97152	−0.23693	−4.10050
1.32	0.96872	0.24818	3.90335	1.82	0.96911	−0.24663	−3.92937
1.33	0.97115	0.23848	4.07231	1.83	0.96659	−0.25631	−3.77118
1.34	0.97348	0.22875	4.25562	1.84	0.96398	−0.26596	−3.62450
1.35	0.97572	0.21901	4.45523	1.85	0.96127	−0.27559	−3.48806
1.36	0.97786	0.20924	4.67344	1.86	0.95847	−0.28519	−3.36083
1.37	0.97991	0.19945	4.91306	1.87	0.95557	−0.29476	−3.24188
1.38	0.98185	0.18964	5.17744	1.88	0.95257	−0.30430	−3.13039
1.39	0.98370	0.17981	5.47069	1.89	0.94949	−0.31381	−3.02566
1.40	0.98545	0.16997	5.79788	1.90	0.94630	−0.32329	−2.92710
1.41	0.98710	0.16010	6.16537	1.91	0.94302	−0.33274	−2.83414
1.42	0.98865	0.15023	6.58110	1.92	0.93964	−0.34215	−2.74630
1.43	0.99010	0.14033	7.05546	1.93	0.93618	−0.35153	−2.66316
1.44	0.99146	0.13042	7.60182	1.94	0.93262	−0.36087	−2.58433
1.45	0.99271	0.12050	8.23810	1.95	0.92896	−0.37018	−2.50947
1.46	0.99387	0.11057	8.98862	1.96	0.92521	−0.37945	−2.43828
1.47	0.99492	0.10063	9.88740	1.97	0.92137	−0.38868	−2.37049
1.48	0.99588	0.09067	10.98338	1.98	0.91744	−0.39788	−2.30582
1.49	0.99674	0.08771	12.34991	1.99	0.91341	−0.40703	−2.24408
1.50	0.99749	0.07074	14.10142	2.00	0.90930	−0.41615	−2.18504

TABLE 11 : 1 — (*Continued*)

NATURAL SINES, COSINES, AND TANGENTS OF ANGLES IN RADIANS

L/j in radians	Sine $\frac{L}{j}$	Cosine $\frac{L}{j}$	Tangent $\frac{L}{j}$	L/j in radians	Sine $\frac{L}{j}$	Cosine $\frac{L}{j}$	Tangent $\frac{L}{j}$
2.00	0.90930	−0.41615	−2.18504	2.50	0.59847	−0.80114	−0.74703
2.01	0.90509	−0.42522	−2.12853	2.51	0.59043	−0.80709	−0.73155
2.02	0.90079	−0.43425	−2.07437	2.52	0.58233	−0.81295	−0.71632
2.03	0.89641	−0.44323	−2.02242	2.53	0.57417	−0.81873	−0.70129
2.04	0.89193	−0.45218	−1.97252	2.54	0.56596	−0.82444	−0.68647
2.05	0.88736	−0.46107	−1.92456	2.55	0.55769	−0.83005	−0.67186
2.06	0.88270	−0.46992	−1.87841	2.56	0.54936	−0.83559	−0.65744
2.07	0.87797	−0.47873	−1.83396	2.57	0.54097	−0.84104	−0.64322
2.08	0.87313	−0.48748	−1.79112	2.58	0.53253	−0.84641	−0.62917
2.09	0.86822	−0.49619	−1.74977	2.59	0.52405	−0.85169	−0.61530
2.10	0.86319	−0.50485	−1.70984	2.60	0.51550	−0.85689	−0.60160
2.11	0.85812	−0.51345	−1.67127	2.61	0.50691	−0.86200	−0.58806
2.12	0.85294	−0.52201	−1.63395	2.62	0.49827	−0.86703	−0.57468
2.13	0.84768	−0.53051	−1.59785	2.63	0.48957	−0.87197	−0.56145
2.14	0.84233	−0.53896	−1.56287	2.64	0.48082	−0.87682	−0.54837
2.15	0.83690	−0.54736	−1.52898	2.65	0.47204	−0.88158	−0.53544
2.16	0.83138	−0.55570	−1.49610	2.66	0.46319	−0.88626	−0.52264
2.17	0.82579	−0.56398	−1.46419	2.67	0.45431	−0.89085	−0.50997
2.18	0.82010	−0.57221	−1.43321	2.68	0.44538	−0.89534	−0.49744
2.19	0.81434	−0.58039	−1.40310	2.69	0.43640	−0.89975	−0.48502
2.20	0.80849	−0.58850	−1.37382	2.70	0.42738	−0.90407	−0.47273
2.21	0.80258	−0.59656	−1.34534	2.71	0.41831	−0.90830	−0.46055
2.22	0.79657	−0.60455	−1.31761	2.72	0.40922	−0.91244	−0.44849
2.23	0.79048	−0.61249	−1.29060	2.73	0.40007	−0.91648	−0.43653
2.24	0.78432	−0.62036	−1.26429	2.74	0.39089	−0.92044	−0.42467
2.25	0.77807	−0.62817	−1.23863	2.75	0.38167	−0.92430	−0.41292
2.26	0.77175	−0.63592	−1.21360	2.76	0.37240	−0.92807	−0.40126
2.27	0.76536	−0.64361	−1.18916	2.77	0.36310	−0.93175	−0.38970
2.28	0.75888	−0.65123	−1.16531	2.78	0.35377	−0.93533	−0.37822
2.29	0.75233	−0.65879	−1.14199	2.79	0.34440	−0.93883	−0.36684
2.30	0.74571	−0.66628	−1.11921	2.80	0.33499	−0.94222	−0.35553
2.31	0.73902	−0.67370	−1.09694	2.81	0.32555	−0.94553	−0.34431
2.32	0.73224	−0.68106	−1.07514	2.82	0.31608	−0.94873	−0.33316
2.33	0.72539	−0.68834	−1.05381	2.83	0.30658	−0.95185	−0.32209
2.34	0.71847	−0.69556	−1.03292	2.84	0.29704	−0.95487	−0.31109
2.35	0.71148	−0.70271	−1.01247	2.85	0.28748	−0.95779	−0.30014
2.36	0.70441	−0.70979	−0.99242	2.86	0.27788	−0.96061	−0.28928
2.37	0.69728	−0.71680	−0.97276	2.87	0.26827	−0.96335	−0.27847
2.38	0.69007	−0.72374	−0.95349	2.88	0.25862	−0.96598	−0.26773
2.39	0.68281	−0.73060	−0.93457	2.89	0.24895	−0.96852	−0.25704
2.40	0.67547	−0.73739	−0.91602	2.90	0.23925	−0.97096	−0.24641
2.41	0.66806	−0.74411	−0.89779	2.91	0.22952	−0.97330	−0.23583
2.42	0.66058	−0.75076	−0.87989	2.92	0.21979	−0.97555	−0.22529
2.43	0.65304	−0.75732	−0.86230	2.93	0.21002	−0.97770	−0.21481
2.44	0.64544	−0.76383	−0.84502	2.94	0.20023	−0.97975	−0.20437
2.45	0.63777	−0.77023	−0.82801	2.95	0.19042	−0.98170	−0.19397
2.46	0.63003	−0.77657	−0.81130	2.96	0.18060	−0.98356	−0.18362
2.47	0.62224	−0.78283	−0.79485	2.97	0.17076	−0.98531	−0.17330
2.48	0.61438	−0.78901	−0.77866	2.98	0.16089	−0.98697	−0.16301
2.49	0.60646	−0.79512	−0.76272	2.99	0.15101	−0.98853	−0.15276
2.50	0.59847	−0.80114	−0.74703	3.00	0.14112	−0.98999	−0.14254

TABLE 11 : 1 — (*Concluded*)

NATURAL SINES, COSINES, AND TANGENTS OF ANGLES IN RADIANS

L/j in radians	Sine $\frac{L}{j}$	Cosine $\frac{L}{j}$	Tangent $\frac{L}{j}$	L/j in radians	Sine $\frac{L}{j}$	Cosine $\frac{L}{j}$	Tangent $\frac{L}{j}$
3.00	0.14112	−0.98999	−0.14254	3.25	−0.10820	−0.99413	0.10883
3.01	0.13121	−0.99135	−0.13235	3.26	−0.11814	−0.99300	0.11896
3.02	0.12129	−0.99262	−0.12219	3.27	−0.12806	−0.99177	0.12912
3.03	0.11136	−0.99378	−0.11206	3.28	−0.13797	−0.99044	0.13930
3.04	0.10142	−0.99484	−0.10195	3.29	−0.14787	−0.98901	0.14951
3.05	0.09146	−0.99581	−0.09185	3.30	−0.15774	−0.98748	0.15975
3.06	0.08150	−0.99667	−0.08177	3.31	−0.16761	−0.98585	0.17002
3.07	0.07153	−0.99744	−0.07171	3.32	−0.17746	−0.98412	0.18033
3.08	0.06155	−0.99810	−0.06167	3.33	−0.18729	−0.98230	0.19067
3.09	0.05156	−0.99867	−0.05164	3.34	−0.19711	−0.98039	0.20105
3.10	0.04159	−0.99913	−0.04162	3.35	−0.20690	−0.97836	0.21148
3.11	0.03159	−0.99950	−0.03161	3.36	−0.21668	−0.97624	0.22195
3.12	0.02160	−0.99977	−0.02160	3.37	−0.22643	−0.97403	0.23246
3.13	0.01160	−0.99993	−0.01160	3.38	−0.23616	−0.97172	0.24303
3.14	0.00160	−1.00000	−0.00160	3.39	−0.24587	−0.96930	0.25365
3.15	−0.00841	−0.99997	0 00841	3.40	−0.25555	−0.96680	0.26431
3.16	−0.01841	−0.99983	0.01841	3.41	−0.26520	−0.96419	0.27504
3.17	−0.02840	−0.99960	0.02841	3.42	−0.24782	−0.96149	0.28583
3.18	−0.03840	−0.99926	0.03843	3.43	−0.28443	−0.95870	0.29668
3.19	−0.04839	−0.99883	0.04845	3.44	−0.29400	−0.95581	0.30759
3.20	−0.05838	−0.99830	0.05848	3.45	−0.30354	−0.95282	0.31857
3.21	−0.06936	−0.99766	0.06852	3.46	−0.31306	−0.94974	0.32962
3.22	−0.07833	−0.99693	0.07857	3.47	−0.32254	−0.94656	0.34074
3.23	−0.08829	−0.99609	0.08864	3.48	−0.33199	−0.94328	0.35195
3.24	−0.09825	−0.99516	0.09873	3.49	−0.34141	−0.93992	0.36322
3.25	−0.10820	−0.99413	0.10883	3.50	−0.35077	−0.93646	0.37459

11 : 4. Numerical Example — Design a rear upper spar for the airplane framework shown in Fig. 7 : 10, assuming the spar to be a spruce beam of *I*-section and continuous between wing tips and with a center height of not over 4.00 in. From Fig. 7 : 13 the average load in the outer bay of this spar due to its being a member of the drag truss is 1/2 (355 + 163) = 259, say 260 lb., compression. The corresponding load in the center section is 355 lb. From the analysis of this framework in Art. 7 : 6 (see Chapter VII) the axial load in this spar due to its being a member of the lift truss is 2890 lb. compression in the outer bay, and 2890 + 150 = 3040 lb. compression in the center section. The total axial load in the bay is therefore 260 + 2890 = 3150 lb. and in the center section 355 + 3040 = 3395 lb. From page 97 of Chapter VII, the bending moment at the outer support, M_1, is 1200 in.-lb. for a normal load of 1.0 lb. per in. For the actual load of 9.0 lb. per in., therefore, M_1 = 10,800 in.-lb.

Following usual airplane practice, the spar section at the joints of

the lift and drag trusses will be rectangular except for the bevel of the upper and lower surfaces to conform to the wing contour. Between those sections, it will be routed out to an *I*-section. For a first trial the width of the spar will be assumed to be between one-fourth and one-third of the depth, i.e. between 1.00 and 1.33 in., say 1.25 in. The routing for this first trial will be assumed to cut away about 40 per cent of the rectangle, leaving the section shown in Fig. 11 : 4. Although the actual section is beveled and has fillets between the web and flanges, as shown, the geometrical properties used in the computations will be those of the equivalent section, shown by the dotted lines, which has the same center height and flange thicknesses as the actual section, but in which the bevels and fillets are neglected. The properties of this routed section are then:

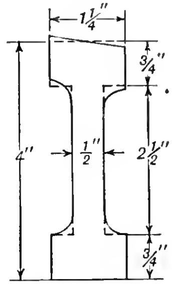


FIG. 11 : 4

$$A = 1.25 \times 4.00 - 0.75 \times 2.50 = 3.125 \text{ in.}^2$$

$$I = \frac{1.25 \times 4.00^3}{12} - \frac{0.75 \times 2.50^3}{12} = 5.69 \text{ in.}^4$$

The properties of the routed section only are used in the three-moment equation because the lengths of the unrouted rectangular sections are small and will have relatively little effect on the deflections of the beam and the secondary bending moments. The computations of bending moments by the aid of the precise formulas may now be made and tabulated as shown below.

Span 1–2 (Outer Bay)

$w_1 = 9.00$

$M_1 = 10{,}800$

$I_1 = 5.69$

$E_1 = 1{,}300{,}000$

$E_1I_1 = 7{,}397{,}000$

$P_1 = 3{,}150$

$j_1^2 = \dfrac{E_1I_1}{P_1} = 2348.25$

$j_1 = 48.46$

$w_1j_1^2 = 21{,}134$

$L_1 = 96.0$

$\dfrac{L_1}{j_1} = 1.981$

$\alpha_1 = 1.7746$ from table 11 : 3

$\beta_1 = 1.4234$ from table 11 : 3

$\gamma_1 = 1.6517$ from table 11 : 3

Span 2–3 (Center Section)

$w_2 = 9.00$

$M_2 = M_3$ by symmetry

$I_2 = 5.69$

$E_2 = 1{,}300{,}000$

$E_2I_2 = 7{,}397{,}000$

$P_2 = 3395$

$j_2^2 = \dfrac{E_2I_2}{P_2} = 2178.79$

$j_2 = 46.68$

$w_2j_2^2 = 19{,}609$

$L_2 = 48.0$

$\dfrac{L_2}{j_2} = 1.028$

$\alpha_2 = 1.1389$ from table 11 : 3

$\beta_2 = 1.0785$ from table 11 : 3

$\gamma_2 = 1.1185$ from table 11 : 3

Since $I_1 = I_2$ the three-moment equation for this case is

$$M_1L_1\alpha_1 + 2\,M_2(L_1\beta_1 + L_2\beta_2) + M_2L_2\alpha_2 = \frac{w_1L_1^3\gamma_1}{4} + \frac{w_2L_2^3\gamma_2}{4}$$

$$10{,}800 \times 96 \times 1.7746 + 2\,M_2(96 \times 1.4234 + 48 \times 1.0785) + M_2 \times$$

$$48 \times 1.1389 = \frac{9.00 \times 96^3 \times 1.6517}{4} + \frac{9.00 \times 48^3 \times 1.1185}{4}$$

$$1{,}839{,}900 + 376.83\,M_2 + 54.67\,M_2 = 3{,}287{,}900 + 278{,}300$$

$$M_2 = M_3 = \frac{1{,}726{,}300}{431.50} = 4001 \text{ in.-lb.}$$

The next step is to determine the magnitudes of the maximum bending moments in the outer bay and center section. In the latter the point of zero slope of the moment curve — the mathematical maximum or minimum — will be at the center of the span, as the loading is symmetrical. The loading on the outer bay, however, is not symmetrical, so the location of the point of maximum bending moment must be computed from the formula

$$\tan\frac{x_1}{j_1} = \frac{D_2 - D_1\cos\dfrac{L_1}{j_1}}{D_1\sin\dfrac{L_1}{j_1}}$$

where

$$D_1 = M_1 - w_1j_1^2 = 10{,}800 - 21{,}134 = -10{,}334$$
$$D_2 = M_2 - w_1j_1^2 = 4{,}001 - 21{,}134 = -17{,}133$$

and from Table 11 : 1.

$$\cos\frac{L_1}{j_1} = \cos 1.981 = -0.39810$$

$$\sin\frac{L_1}{j_1} = \sin 1.981 = 0.91704$$

then

$$\tan\frac{x_1}{j_1} = \frac{-17{,}133 - (-10{,}334)\,(-0.39810)}{(-10{,}334)\,(0.91704)} = 2.242$$

$$\frac{x_1}{j_1} = \tan^{-1} 2.242 = 1.151$$

The point of maximum moment is then $x_1 = 48.46 \times 1.151 = 55.78$ in. from R_1 at the outer strut.

The magnitude of the maximum moment in the bay will be

$$M_{1-2\ \max} = \frac{D_1}{\cos\dfrac{x_1}{j_1}} + w_1j_1^2 = \frac{-10{,}334}{0.40739} + 21{,}134 = -\,4{,}202 \text{ in.-lb.}$$

In the center section, $\frac{x_2}{j_2} = \frac{L_2}{2\,j_2} = 0.514$

$$M_{2-3\ \max} = \frac{4{,}001 - 19{,}609}{0.87077} + 19{,}609 = +1{,}685 \text{ in.-lb.}$$

The investigation of this spar section including the computation of the allowable stresses and the margins of safety under the loads computed above, is completed in Art. 12 : 2.

11 : 5. Formulas for Other Loadings — Formulas for both single spans and continuous beams under several loading conditions will now be given. These include the cases for a concentrated and a uniformly varying transverse load with axial compression and a uniform transverse load with axial tension. As it is more conservative to neglect the secondary effects of an axial tension, the tendency of which is to reduce the bending moments, and to design for the fiber stresses produced by the bending moments caused by the transverse loads acting alone plus the stresses due to the axial tension, that practice is recommended, and the formulas for other cases including axial tension are omitted.

All of the formulas given here were derived by the method used above for the uniformly distributed load, and a comparative study of them yields interesting and important results, which will assist the engineer in writing down expressions for loadings not covered specifically.

In every case with axial compression the formula for bending moment in a single span is in the form

$$M = C_1 \sin\frac{x}{j} + C_2 \cos\frac{x}{j} + f(w)$$

where $f(w)$ is a term containing w and possibly j, x, and L, but neither the axial load P nor the bending moments M_1 and M_2. Its form depends on the character of the lateral load. The constants C_1 and C_2 which also vary with the type of loading may be divided into "moment terms" containing M_1 and M_2 and "load terms" containing w or W. The moment terms are the same for all cases; only the load terms depend on the type of side load. As a result of this fact, a very simple method can be used to write the formulas for the bending moments caused by an axial compression and any combination of the lateral loads for which the formulas are given. This is to write the formula for any one of the loads in question and add to it the "load terms" of the constants C_1 and C_2 for the other loads, each multiplied by $\sin\frac{x}{j}$ or $\cos\frac{x}{j}$ as the case may be. Study of the formula for a series of concentrated

loads given on page 207 will show that it could be obtained in this manner, and the procedure may also be used when uniform or uniformly distributed loads are involved. The same principle can be used also for writing equations for the slope of the elastic curve and for the deflection.

No simple system like that just described for writing down formulas for the bending moment at any point has been devised for determining the location of the point of maximum moment or the amount of that moment. The best way to accomplish this for a complex loading, and even for rather simple loadings like the uniformly varying load, is to plot the curve of bending moment using sufficient points so that the maximum moment and its location can be determined with the desired precision.

The specific formulas for the more common cases of single spans subjected to combined bending and axial compression follow. In all of these formulas and those for combined bending and tension and the three-moment equations for cases involving axial load, the following nomenclature and conventions for signs are used:

w = intensity of distributed lateral load in lb. per in., positive when acting upward.

W = magnitude of concentrated lateral load in lb., positive when acting upward.

S = shear, in lb., positive when the algebraic sum of all forces acting on the beam to the left of the section considered is positive.

M = bending moment in in.-lb., positive when it tends to cause compression in the upper fibers of the part of the beam to the right of the section.

i = slope of elastic curve of the beam in radians, positive when the tangent rises from left to right.

δ = deflection in in., positive when the deflected position of a point is above the original position.

P = axial load, lb.

E = modulus of elasticity of the material, lb. per sq. in.

I = moment of inertia of the section in in.^{4}

$j = \sqrt{\dfrac{EI}{P}}$

L = length of span between supports in inches.

x = distance to a section from the left end of the span in which it is located, in inches.

Where two or more quantities of any class appear in a formula, they are differentiated by suitable subscripts.

CASE A. NO SIDE LOADS, UNEQUAL END MOMENTS.

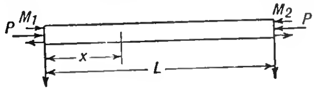


FIG. 11 : 5

$$S = \frac{M_2 - M_1 \cos\frac{L}{j}}{j \sin\frac{L}{j}} \cos\frac{x}{j} - \frac{M_1}{j} \sin\frac{x}{j}$$

$$M = \frac{M_2 - M_1 \cos\frac{L}{j}}{\sin\frac{L}{j}} \sin\frac{x}{j} + M_1 \cos\frac{x}{j}$$

$$i = \frac{1}{P}\left[\frac{M_2 - M_1}{L} - \frac{M_2 - M_1 \cos\frac{L}{j}}{j \sin\frac{L}{j}} \cos\frac{x}{j} + \frac{M_1}{j} \sin\frac{x}{j}\right]$$

$$\delta = \frac{1}{P}\left[M_1 + \frac{M_2 - M_1}{L} x - \frac{M_2 - M_1 \cos\frac{L}{j}}{\sin\frac{L}{j}} \sin\frac{x}{j} - M_1 \cos\frac{x}{j}\right]$$

At point of maximum moment $\tan\frac{x}{j} = \dfrac{M_2 - M_1 \cos\frac{L}{j}}{M_1 \sin\frac{L}{j}}$

$$M_{\max} = \frac{M_1}{\cos\frac{x}{j}}$$

CASE B. UNIFORM SIDE LOAD, UNEQUAL END MOMENTS.

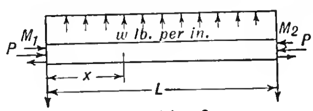


FIG. 11 : 6

$$D_1 = M_1 - wj^2 \qquad D_2 = M_2 - wj^2$$

$$S = \frac{D_2 - D_1 \cos\frac{L}{j}}{j \sin\frac{L}{j}} \cos\frac{x}{j} - \frac{D_1}{j} \sin\frac{x}{j}$$

$$M = \frac{D_2 - D_1 \cos\frac{L}{j}}{\sin\frac{L}{j}} \sin\frac{x}{j} + D_1 \cos\frac{x}{j} + wj^2$$

$$i = \frac{1}{P}\left[\frac{M_2 - M_1}{L} - \frac{wL}{2} + wx - \frac{D_2 - D_1 \cos\frac{L}{j}}{j \sin\frac{L}{j}} \cos\frac{x}{j} + \frac{D_1}{j} \sin\frac{x}{j}\right]$$

$$\delta = \frac{1}{P}\left[M_1 + \frac{M_2 - M_1}{L}x - \frac{wLx}{2} + \frac{wx^2}{2} - \frac{D_2 - D_1 \cos\frac{L}{j}}{\sin\frac{L}{j}} \sin\frac{x}{j} - D_1 \cos\frac{x}{j} - wj^2\right]$$

At point of maximum moment $\tan\frac{x}{j} = \dfrac{D_2 - D_1 \cos\frac{L}{j}}{D_1 \sin\frac{L}{j}}$

$$M_{\max} = \frac{D_1}{\cos\frac{x}{j}} + wj^2$$

Case B′. Uniform Side Load, Pin Ends.

$$S = \frac{wj\left(\cos\frac{L}{j} - 1\right)}{\sin\frac{L}{j}} \cos\frac{x}{j} + wj \sin\frac{x}{j}$$

$$M = \frac{wj^2\left(\cos\frac{L}{j} - 1\right)}{\sin\frac{L}{j}} \sin\frac{x}{j} + wj^2\left(1 - \cos\frac{x}{j}\right)$$

$$i = \frac{1}{P}\left[wx - \frac{wL}{2} + \frac{wj\left(1 - \cos\frac{L}{j}\right)}{\sin\frac{L}{j}} \cos\frac{x}{j} - wj \sin\frac{x}{j}\right]$$

$$\delta = \frac{1}{P}\left[\frac{wx^2}{2} - \frac{wLx}{2} + \frac{wj^2\left(1 - \cos\frac{L}{j}\right)}{\sin\frac{L}{j}} \sin\frac{x}{j} - wj^2\left(1 - \cos\frac{x}{j}\right)\right]$$

Moment is maximum at $x = 0.5\,L$

$$M_{\max} = wj^2\left[1 - \frac{1}{\cos\frac{L}{2j}}\right] = wj^2\left(1 - \sec\frac{L}{2j}\right)$$

Approximate formulas for $M_{\max}$.

Johnson formula

$$M_{\max} = \frac{M_0}{1 - \dfrac{PL^2}{10\,EI}} \quad \text{or} \quad \frac{M_0}{1 - \dfrac{5\,PL^2}{48\,EI}}$$

Perry's formula

$$M_{\max} = \frac{M_0 Q}{Q - P}$$

where

$$M_0 = -\frac{wL^2}{8} \qquad Q = \frac{\pi^2 EI}{L^2}$$

Case C. Uniformly Varying Load Increasing to the Right, Unequal End Moments.

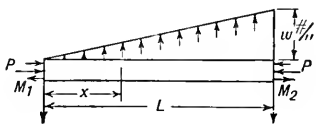


Fig. 11 : 7

$$C_1 = \frac{M_2 - wj^2 - M_1 \cos\dfrac{L}{j}}{\sin\dfrac{L}{j}} \qquad C_2 = M_1$$

$$S = \frac{C_1}{j}\cos\frac{x}{j} - \frac{C_2}{j}\sin\frac{x}{j} + \frac{wj^2}{L}$$

$$M = C_1 \sin\frac{x}{j} + C_2 \cos\frac{x}{j} + \frac{wj^2}{L}x$$

$$i = \frac{1}{P}\left[\frac{M_2 - M_1}{L} - \frac{wL}{6} + \frac{wx^2}{2\,L} - \frac{C_1}{j}\cos\frac{x}{j} + \frac{C_2}{j}\sin\frac{x}{j} - \frac{wj^2}{L}\right]$$

$$\delta = \frac{1}{P}\left[M_1 + \frac{M_2 - M_1}{L}x - \frac{wLx}{6} + \frac{wx^3}{6\,L} - C_1 \sin\frac{x}{j} - C_2 \cos\frac{x}{j} - \frac{wj^2x}{L}\right]$$

At point of maximum moment

$$\tan\frac{x}{j} = \frac{C_1}{C_2} - \frac{(C_1^2 + C_2^2)wj^3}{C_1C_2wj^3 \pm C_2^2\sqrt{L^2(C_1^2 + C_2^2) - (wj^3)^2}}$$

CASE D. UNIFORMLY VARYING LOAD INCREASING FROM w AT THE LEFT TO $w + kw$ AT THE RIGHT SUPPORT. UNEQUAL END MOMENTS.

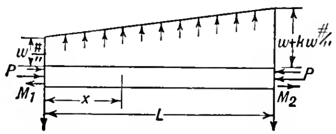


FIG. 11 : 8

$$C_1 = \frac{M_2 - (1 + k)wj^2 - (M_1 - wj^2)\cos\frac{L}{j}}{\sin\frac{L}{j}} \qquad C_2 = M_1 - wj^2$$

$$S = \frac{C_1}{j}\cos\frac{x}{j} - \frac{C_2}{j}\sin\frac{x}{j} + \frac{kwj^2}{L}$$

$$M = C_1 \sin\frac{x}{j} + C_2\cos\frac{x}{j} + wj^2\left(1 + \frac{kx}{L}\right)$$

$$i = \frac{1}{P}\left[\frac{M_2 - M_1}{L} - \frac{wL}{2} - \frac{kwL}{6} + wx + \frac{kwx^2}{2L} - \frac{C_1}{j}\cos\frac{x}{j} + \frac{C_2}{j}\sin\frac{x}{j} - \frac{kwj^2}{L}\right]$$

$$\delta = \frac{1}{P}\left[M_1 + \frac{M_2 - M_1}{L}x - \frac{wLx}{2} - \frac{kwLx}{6} + \frac{wx^2}{2} + \frac{kwx^3}{6L} - C_1\sin\frac{x}{j} - C_2\cos\frac{x}{j} - wj^2\left(1 + \frac{kx}{L}\right)\right]$$

The expression for $\tan\frac{x}{j}$ at the section of maximum moment is so cumbersome as to be impracticable. In order to obtain the maximum moment it will be best to compute the moments at three or four points in the span, plot the results, and obtain the maximum value from a smooth curve through the points plotted.

CASE E. SINGLE CONCENTRATED LOAD, UNEQUAL END MOMENTS.

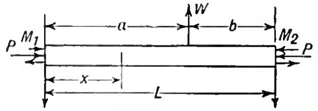


FIG. 11 : 9

$$C_1 = \frac{M_2}{\sin\frac{L}{j}} - \frac{M_1 - Wj\sin\frac{a}{j}}{\tan\frac{L}{j}} - Wj\cos\frac{a}{j} = C_3 - Wj\cos\frac{a}{j}$$

$$C_2 = M_1$$

$$C_3 = \frac{M_2}{\sin \frac{L}{j}} - \frac{M_1 - Wj \sin \frac{a}{j}}{\tan \frac{L}{j}}$$

$$C_4 = M_1 - Wj \sin \frac{a}{j}$$

where $x < a$

$$S = \frac{C_1}{j} \cos \frac{x}{j} - \frac{C_2}{j} \sin \frac{x}{j}$$

$$M = C_1 \sin \frac{x}{j} + C_2 \cos \frac{x}{j}$$

$$i = \frac{1}{P}\left[\frac{M_2 - M_1}{L} - \frac{Wb}{L} - \frac{C_1}{j} \cos \frac{x}{j} + \frac{C_2}{j} \sin \frac{x}{j}\right]$$

$$\delta = \frac{1}{P}\left[M_1 + \frac{M_2 - M_1}{L} x - \frac{Wbx}{L} - C_1 \sin \frac{x}{j} - C_2 \cos \frac{x}{j}\right]$$

where $x > a$

$$S = \frac{C_3}{j} \cos \frac{x}{j} - \frac{C_4}{j} \sin \frac{x}{j}$$

$$M = C_3 \sin \frac{x}{j} + C_4 \cos \frac{x}{j}$$

$$i = \frac{1}{P}\left[\frac{M_2 - M_1}{L} + \frac{Wa}{L} - \frac{C_3}{j} \cos \frac{x}{j} + \frac{C_4}{j} \sin \frac{x}{j}\right]$$

$$\delta = \frac{1}{P}\left[M_1 + \frac{M_2 - M_1}{L} x + \frac{Wa}{L}(x - L) - C_3 \sin \frac{x}{j} - C_4 \cos \frac{x}{j}\right]$$

at point of maximum moment $\tan \frac{x}{j} = \frac{C_1}{C_2}$ or $\frac{C_3}{C_4}$

$$M_{\text{max}} = \frac{C_1^2 + C_2^2}{C_2} \cos \frac{x}{j} \quad \text{or} \quad \frac{C_3^2 + C_4^2}{C_4} \cos \frac{x}{j}$$

To find the maximum moment, compute x from the two expressions for $\tan x/j$. If C_1/C_2 gives a value of x less than a, or C_3/C_4 gives a value of x between a and L, that value of x is the abscissa of the point of maximum moment and should be used in the corresponding expression given above for M_{max}. If neither C_1/C_2 nor C_3/C_4 gives a value of x within the range of the corresponding moment equation, the maximum moment is either at one of the supports or at the load. To find the maximum moment in this case one of the formulas for moment at any point on the span must be used, as the formulas for maximum moment will give incorrect results.

CASE F. TWO SYMMETRICALLY LOCATED EQUAL CONCENTRATED LOADS, EQUAL END MOMENTS.

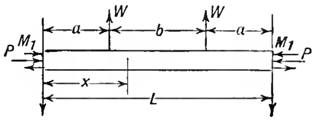


FIG. 11 : 10

$$C_1 = M_1 \tan \frac{L}{2j} - Wj \frac{\cos \frac{b}{2j}}{\cos \frac{L}{2j}}$$

$$C_2 = M_1$$

$$C_3 = M_1 \tan \frac{L}{2j} - Wj \sin \frac{a}{j} \tan \frac{L}{2j} = C_4 \tan \frac{L}{2j}$$

$$C_4 = M_1 - Wj \sin \frac{a}{j}$$

where $x < a$

$$S = \frac{C_1}{j} \cos \frac{x}{j} - \frac{C_2}{j} \sin \frac{x}{j}$$

$$M = C_1 \sin \frac{x}{j} + C_2 \cos \frac{x}{j}$$

$$i = \frac{1}{P}\left[-\frac{C_1}{j} \cos \frac{x}{j} + \frac{C_2}{j} \sin \frac{x}{j} - W \right]$$

$$\delta = \frac{1}{P}\left[- C_1 \sin \frac{x}{j} - C_2 \cos \frac{x}{j} - Wx + M_1 \right]$$

when $a < x < (a + b)$

$$S = \frac{C_3}{j} \cos \frac{x}{j} - \frac{C_4}{j} \sin \frac{x}{j}$$

$$M = C_3 \sin \frac{x}{j} + C_4 \cos \frac{x}{j}$$

$$i = \frac{1}{P}\left[-\frac{C_3}{j} \cos \frac{x}{j} + \frac{C_4}{j} \sin \frac{x}{j} \right]$$

$$\delta = \frac{1}{P}\left[- C_3 \sin \frac{x}{j} - C_4 \cos \frac{x}{j} - Wa + M_1 \right]$$

when $x = a$

$$M_a = \left(M_1 - Wj \sin \frac{a}{j} \right) \frac{\cos \frac{b}{2j}}{\cos \frac{L}{2j}}$$

$$i_a = -\frac{1}{Pj} \left(M_1 - Wj \sin \frac{a}{j} \right) \frac{\sin \frac{b}{2j}}{\cos \frac{L}{2j}}$$

$$\delta_a = \frac{1}{P}\left[M_1 - Wa - \left(M_1 - Wj\sin\frac{a}{j}\right)\frac{\cos\frac{b}{2j}}{\cos\frac{L}{2j}}\right]$$

Where $x = 0.5\ L$

$$M_{\max} = \frac{M_1}{\cos\frac{L}{2j}} - \frac{Wj\sin\frac{a}{j}}{\cos\frac{L}{2j}}$$

$$i_{L/2} = 0$$

$$\delta_{\max} = \frac{1}{P}\left[M_1 - Wa - \frac{M_1 - Wj\sin\frac{a}{j}}{\cos\frac{L}{2j}}\right]$$

Case G. Series of Concentrated Loads, Unequal End Moments.

$$M = \frac{M_2 - M_1\cos\frac{L}{j}}{\sin\frac{L}{j}}\sin\frac{x}{j} + M_1\cos\frac{x}{j} - \frac{j\sin\frac{x}{j}}{\sin\frac{L}{j}}\sum W\sin\frac{L-a}{j}$$
$$- \frac{j\sin\frac{L-x}{j}}{\sin\frac{L}{j}}\sum W\sin\frac{a}{j}$$

when the $\sum W\sin\frac{L-a}{j}$ terms apply to the loads to the right of the section, or $a > x$, and the $\sum W\sin\frac{a}{j}$ terms apply to the loads to the left, or $a < x$.

Case H. General Expression for Beams where the Axial Load is Compression.

The formula for the moment at any point due to a series of concentrated loads, Case G, may be written as a general expression for the moment if it is assumed that between $x = a$ and $x = a + da$ there is a small force wda. The effect of all such forces between $x = 0$ and $x = L$ may be found by the substitution of the appropriate values in the equation of Case G above and summing, or, since we are dealing with infinitesimals in this case, integrating. Hence

$$M = \frac{M_2 - M_1\cos\frac{L}{j}}{\sin\frac{L}{j}}\sin\frac{x}{j} + M_1\cos\frac{x}{j} - \frac{j\sin\frac{x}{j}}{\sin\frac{L}{j}}\int_x^L w\sin\frac{L-a}{j}\,da$$
$$- \frac{j\sin\frac{L-x}{j}}{\sin\frac{L}{j}}\int_0^x w\sin\frac{a}{j}\,da$$

If we let $a + b = L$, then $L - a = b$, $da = -db$, and the limits for b which correspond to x and L for a become $L - x$ and 0. Whence

$$M = \frac{M_2 - M_1 \cos \frac{L}{j}}{\sin \frac{L}{j}} \sin \frac{x}{j} + M_1 \cos \frac{x}{j} + \frac{j \sin \frac{x}{j}}{\sin \frac{L}{j}} \int_{L-x}^{0} w \sin \frac{b}{j}\, db$$

$$- \frac{j \sin \frac{L - x}{j}}{\sin \frac{L}{j}} \int_0^x w \sin \frac{a}{j}\, da.$$

The terms in b depend on the forces to the right of the section, those in a depend on the forces to the left, so that two expressions must be derived for w, one in terms of b and one in terms of a. If w is constant, the expressions are easily integrated and reduce to those already developed for a uniformly distributed load.

The formulas for a single span subjected to a uniform side load, unequal end moments, and axial tension are as follows:

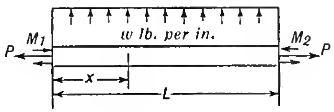


FIG. 11 : 11

$$D_1 = M_1 + wj^2 \qquad\qquad D_2 = M_2 + wj^2$$

$$S = \frac{D_2 - D_1 \cosh \frac{L}{j}}{j \sinh \frac{L}{j}} \cosh \frac{x}{j} + \frac{D_1}{j} \sinh \frac{x}{j}$$

$$M = \frac{D_2 - D_1 \cosh \frac{L}{j}}{\sinh \frac{L}{j}} \sinh \frac{x}{j} + D_1 \cosh \frac{x}{j} - wj^2$$

$$i = \frac{1}{P}\left[\frac{D_2 - D_1 \cosh \frac{L}{j}}{j \sinh \frac{L}{j}} \cosh \frac{x}{j} + \frac{D_1}{j} \sinh \frac{x}{j} - \frac{M_2 - M_1}{L} + \frac{wL}{2} - wx\right]$$

$$\delta = \frac{1}{P}\left[\frac{D_2 - D_1 \cosh \frac{L}{j}}{\sinh \frac{L}{j}} \sinh \frac{x}{j} + D_1 \cosh \frac{x}{j} - wj^2 - M_1 - \frac{M_2 - M_1}{L} x \right.$$

$$\left. + \frac{wLx}{2} - \frac{wx^2}{2}\right]$$

At point of maximum moment

$$\tanh\frac{x}{j} = \frac{D_1 \cosh\frac{L}{j} - D_2}{D_1 \sinh\frac{L}{j}}$$

$$M_{\max} = \frac{D_1}{\cosh\frac{x}{j}} - wj^2$$

The expressions for a pin-ended strut or for a restrained strut with no transverse load may readily be obtained from the above formulas.

Three-moment Equations. — The form of the left side of the three-moment equation when there is an axial load present is similar to that for the case of no axial load except that each term is modified by a coefficient to provide for the effect of the axial load. When the axial load causes compression in the span the left side of the equation is

$$\frac{M_1 L_1 \alpha_1}{I_1} + 2\, M_{-2}\frac{L_1}{I_1}\beta_1 + 2\, M_{+2}\frac{L_2}{I_2}\beta_2 + \frac{M_3 L_2 \alpha_2}{I_2}$$

When the axial load causes tension in the span the left side of the equation is

$$\frac{M_1 L_1 \alpha_{h1}}{I_1} + 2\, M_{-2}\frac{L_1}{I_1}\beta_{h1} + 2\, M_{+2}\frac{L_2}{I_2}\beta_{h2} + \frac{M_3 L_2 \alpha_{h2}}{I_2}$$

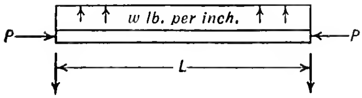
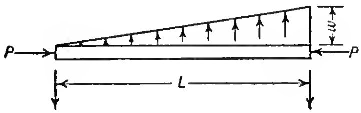
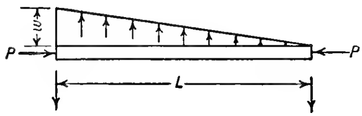
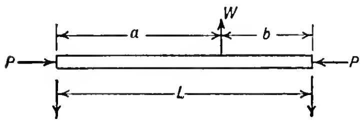
Values of α, β, α_h, and β_h are given in Tables 11 : 3 and 11 : 4.

Attention is called to the fact that the α and β values for the case of axial compression are dependent on the common circular functions, whereas those for axial tension are based upon the hyperbolic functions. The subscript, h, is therefore used to differentiate between the two sets of functions and care should be taken to use the proper values in each case.

The terms on the right side of the equation depend on the type of loading on the spans. They are similar to those used when there is no axial load except for the coefficients which provide for such load. They are listed in Table 11 : 2, two sets of terms being given, one for loads on the left-hand bay, the other for loads on the right. In case the load in any bay is composed of a series of loads, the right side of the three-moment equation may be built up by taking the terms for the component parts from Table 11 : 2 in the same way as for the cases with no axial load. Care must be taken that the correct values of the coefficients are used depending on whether the axial load is tensile or compressive.

TABLE 11 : 2

Loading Terms for Right Side of Precise Three-moment Equation

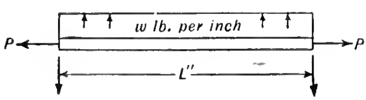
TYPE OF LOADING	IN LEFT BAY	IN RIGHT BAY
	$+\dfrac{w_1 L_1^3 \gamma_1}{4\,I_1}$	$+\dfrac{w_2 L_2^3 \gamma_2}{4\,I_2}$
	$+\dfrac{2\,w_1 L_1 j_1^2(\beta_1 - 1)}{I_1}$	$+\dfrac{w_2 L_2 j_2^2(\alpha_2 - 1)}{I_2}$
	$+\dfrac{w_1 L_1 j_1^2(\alpha_1 - 1)}{I_1}$	$+\dfrac{2\,w_2 L_2 j_2^2(\beta_2 - 1)}{I_2}$
	$+\dfrac{6\,W_1 j_1^2}{I_1}\left[\dfrac{\sin\frac{a_1}{j_1}}{\sin\frac{L_1}{j_1}} - \dfrac{a_1}{L_1}\right]$	$+\dfrac{6\,W_2 j_2^2}{I_2}\left[\dfrac{\sin\frac{b_2}{j_2}}{\sin\frac{L_2}{j_2}} - \dfrac{b_2}{L_2}\right]$

GENERAL EXPRESSION WITH AXIAL COMPRESSIVE LOADS:

$$\frac{L_1}{I_1}\int\left[\left(\frac{x_1}{L_1}\right)^3 - \frac{x_1}{L_1}\right]w_1\,dx \qquad + \frac{L_2}{I_2}\int\left[\left(\frac{L_2 - x_2}{L_2}\right)^3 - \frac{L_2 - x_2}{L_2}\right]w_2\,dx$$

$$+\frac{6\,j_1^2}{I_1}\int\left[\frac{\sin\frac{x_1}{j_1}}{\sin\frac{L_1}{j_1}} - \frac{x_1}{L_1}\right]w_1\,dx \qquad + \frac{6\,j_2^2}{I_2}\int\left[\frac{\sin\dfrac{L_2 - x_2}{j_2}}{\sin\dfrac{L_2}{j_2}} - \frac{L_2 - x_2}{L_2}\right]w_2\,dx$$

UNIFORMLY DISTRIBUTED LOAD WITH AXIAL TENSION:

	$+\dfrac{w_1 L_1^3 \gamma_{h1}}{4\,I_1}$	$+\dfrac{w_2 L_2^3 \gamma_{h2}}{4\,I_2}$

The terms providing for deflection of the supports are identical with those for the cases of no axial load as given in Art. 4 : 1. These terms are added to the right side of the equation in the same way as they appear in Formula 4 : 1 in that article. They are not modified by coefficients to provide for the effect of the axial loads.

The expressions for the bending moment, etc., on an axially loaded beam differ considerably in form from those for beams with the same types of lateral load but no axial load. In many cases, if the axial load and therefore L/j is zero, the formulas reduce to the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Such forms can be evaluated by the standard method described in works on differential calculus, and when this is done will give expressions that are identical with the usual formulas for beams without axial load. Thus the coefficients of the three-moment equation for a beam with a uniformly distributed lateral load all become equal to unity when $L/j = 0$, making this equation the same as that derived in Chapter IV.

The formulas of this article are all based upon the assumption that the moment of inertia of the strut or continuous beam is a constant throughout each span, though it may differ as between spans. If the moment of inertia varies, they will give incorrect results, though if the variation is small or extends over but a small part of the span, they will give results of sufficient precision. For more precise work, the graphical method described in Art. 11 : 13 must be used.

11:6. Tables of Functions for Use in Combined Stress Computations—

TABLE 11:3

FUNCTIONS FOR THREE-MOMENT EQUATIONS — α, β, AND γ FUNCTIONS FOR AXIAL COMPRESSION

$$\alpha = \frac{6\ (L/j \operatorname{cosec} L/j - 1)}{(L/j)^2}$$

$$\beta = \frac{3\ (1 - L/j \cot L/j)}{(L/j)^2}$$

$$\gamma = \frac{3\ (\tan L/2j - L/2j)}{(L/2j)^3}$$

A general relation existing between α, β and γ is $\alpha + 2\beta - 3 = \gamma\ (L/2j)^2$

L/j	α	$\Delta\alpha$	β	$\Delta\beta$	γ	$\Delta\gamma$	L/j
0	1.0000		1.0000		1.0000		0
0.5	1.0300		1.0171		1.0256		0.5
1.00	1.1304		1.0737		1.1113		1.00
		0.0151		0.0085		0.0128	
1.05	1.1455		1.0822		1.1241		1.05
		0.0162		0.0090		0.0138	
1.10	1.1617		1.0912		1.1379		1.10
		0.0175		0.0097		0.0148	
1.15	1.1792		1.1009		1.1527		1.15
		0.0187		0.0105		0.0159	
1.20	1.1979		1.1114		1.1686		1.20
		0.0201		0.0111		0.0170	
1.25	1.2180		1.1225		1.1856		1.25
		0.0216		0.0120		0.0183	
1.30	1.2396		1.1345		1.2039		1.30
		0.0232		0.0128		0.0196	
1.35	1.2628		1.1473		1.2235		1.35
		0.0250		0.0137		0.0210	
1.40	1.2878		1.1610		1.2445		1.40
		0.0268		0.0147		0.0226	
1.45	1.3146		1.1757		1.2671		1.45
		0.0288		0.0158		0.0243	
1.50	1.3434		1.1915		1.2914		1.50
		0.0310		0.0169		0.0260	
1.55	1.3744		1.2084		1.3174		1.55
		0.0334		0.0182		0.0281	
1.60	1.4078		1.2266		1.3455		1.60
		0.0361		0.0196		0.0303	
1.65	1.4439		1.2462		1.3758		1.65
		0.0391		0.0211		0.0327	
1.70	1.4830		1.2673		1.4085		1.70
		0.0422		0.0228		0.0353	
1.75	1.5252		1.2901		1.4438		1.75
		0.0458		0.0246		0.0383	
1.80	1.5710		1.3147		1.4821		1.80
		0.0498		0.0267		0.0416	
1.85	1.6208		1.3414		1.5237		1.85
		0.0542		0.0290		0.0452	
1.90	1.6750		1.3704		1.5689		1.90
		0.0593		0.0316		0.0493	
1.95	1.7343		1.4020		1.6182		1.95
		0.0650		0.0345		0.0540	
2.00	1.7993		1.4365		1.6722		2.00

TABLE 11 : 3

FUNCTIONS FOR THREE-MOMENT EQUATIONS — α, β, AND γ FUNCTIONS FOR AXIAL COMPRESSION — (*Continued*)

L/j	α	$\Delta\alpha$	β	$\Delta\beta$	γ	$\Delta\gamma$	L/j
2.00	1.7993		1.4365		1.6722		2.00
		0.0137		0.0073		0.0114	
2.01	1.8130		1.4438		1.6836		2.01
		0.0140		0.0074		0.0117	
2.02	1.8270		1.4512		1.6953		2.02
		0.0143		0.0075		0.0118	
2.03	1.8413		1.4587		1.7071		2.03
		0.0145		0.0077		0.0121	
2.04	1.8558		1.4664		1.7192		2.04
		0.0148		0.0078		0.0122	
2.05	1.8706		1.4742		1.7314		2.05
		0.0152		0.0080		0.0126	
2.06	1.8858		1.4822		1.7440		2.06
		0.0154		0.0082		0.0128	
2.07	1.9012		1.4904		1.7568		2.07
		0.0156		0.0083		0.0130	
2.08	1.9168		1.4987		1.7698		2.08
		0.0161		0.0084		0.0134	
2.09	1.9329		1.5071		1.7832		2.09
		0.0164		0.0087		0.0135	
2.10	1.9493		1.5158		1.7967		2.10
		0.0168		0.0088		0.0139	
2.11	1.9661		1.5246		1.8106		2.11
		0.0170		0.0090		0.0141	
2.12	1.9831		1.5336		1.8247		2.12
		0.0174		0.0091		0.0145	
2.13	2.0005		1.5427		1.8392		2.13
		0.0179		0.0094		0.0147	
2.14	2.0184		1.5521		1.8539		2.14
		0.0182		0.0095		0.0150	
2.15	2.0366		1.5616		1.8689		2.15
		0.0186		0.0097		0.0154	
2.16	2.0552		1.5713		1.8843		2.16
		0.0189		0.0100		0.0157	
2.17	2.0741		1.5813		1.9000		2.17
		0.0194		0.0101		0.0160	
2.18	2.0935		1.5914		1.9160		2.18
		0.0198		0.0104		0.0163	
2.19	2.1133		1.6018		1.9323		2.19
		0.0203		0.0106		0.0168	
2.20	2.1336		1.6124		1.9491		2.20
		0.0207		0.0109		0.0172	
2.21	2.1543		1.6233		1.9663		2.21
		0.0211		0.0110		0.0174	
2.22	2.1754		1.6343		1.9837		2.22
		0.0218		0.0114		0.0179	
2.23	2.1972		1.6457		2.0016		2.23
		0.0222		0.0115		0.0183	
2.24	2.2194		1.6572		2.0199		2.24
		0.0228		0.0118		0.0187	
2.25	2.2422		1.6690		2.0386		2.25
		0.0232		0.0122		0.0192	
2.26	2.2654		1.6812		2.0578		2.26

TABLE 11 : 3

FUNCTIONS FOR THREE-MOMENT EQUATIONS — α, β, AND γ FUNCTIONS FOR AXIAL COMPRESSION — (*Continued*)

L/j	α	$\Delta\alpha$	β	$\Delta\beta$	γ	$\Delta\gamma$	L/j
2.26	2.2654		1.6812		2.0578		2.26
		0.0237		0.0124		0.0197	
2.27	2.2891		1.6936		2.0775		2.27
		0.0244		0.0126		0.0201	
2.28	2.3135		1.7062		2.0976		2.28
		0.0249		0.0130		0.0205	
2.29	2.3384		1.7192		2.1181		2.29
		0.0256		0.0133		0.0211	
2.30	2.3640		1.7325		2.1392		2.30
		0.0262		0.0136		0.0216	
2.31	2.3902		1.7461		2.1608		2.31
		0.0269		0.0140		0.0222	
2.32	2.4171		1.7601		2.1830		2.32
		0.0277		0.0143		0.0227	
2.33	2.4448		1.7744		2.2057		2.33
		0.0283		0.0147		0.0233	
2.34	2.4731		1.7891		2.2290		2.34
		0.0291		0.0150		0.0239	
2.35	2.5022		1.8041		2.2529		2.35
		0.0298		0.0154		0.0245	
2.36	2.5320		1.8195		2.2774		2.36
		0.0305		0.0159		0.0251	
2.37	2.5625		1.8354		2.3025		2.37
		0.0314		0.0162		0.0259	
2.38	2.5939		1.8516		2.3284		2.38
		0.0323		0.0167		0.0266	
2.39	2.6262		1.8683		2.3550		2.39
		0.0334		0.0171		0.0272	
2.40	2.6596		1.8854		2.3822		2.40
		0.0339		0.0177		0.0281	
2.41	2.6935		1.9031		2.4103		2.41
		0.0352		0.0181		0.0288	
2.42	2.7287		1.9212		2.4391		2.42
		0.0362		0.0186		0.0296	
2.43	2.7649		1.9398		2.4687		2.43
		0.0372		0.0191		0.0306	
2.44	2.8021		1.9589		2.4993		2.44
		0.0382		0.0197		0.0313	
2.45	2.8403		1.9786		2.5306		2.45
		0.0395		0.0203		0.0324	
2.46	2.8798		1.9989		2.5630		2.46
		0.0406		0.0209		0.0334	
2.47	2.9204		2.0198		2.5964		2.47
		0.0420		0.0215		0.0343	
2.48	2.9624		2.0413		2.6307		2.48
		0.0432		0.0222		0.0355	
2.49	3.0056		2.0635		2.6662		2.49
		0.0446		0.0229		0.0365	
2.50	3.0502		2.0864		2.7027		2.50
		0.0461		0.0236		0.0378	
2.51	3.0963		2.1100		2.7405		2.51
		0.0475		0.0243		0.0389	
2.52	3.1438		2.1343		2.7794		2.52

TABLE 11 : 3

FUNCTIONS FOR THREE-MOMENT EQUATIONS — α, β, AND γ FUNCTIONS FOR AXIAL COMPRESSION — (*Continued*)

L/j	α	$\Delta\alpha$	β	$\Delta\beta$	γ	$\Delta\gamma$	L/j
2.52	3.1438		2.1343		2.7794		2.52
		0.0493		0.0252		0.0403	
2.53	3.1931		2.1595		2.8197		2.53
		0.0506		0.0260		0.0415	
2.54	3.2437		2.1855		2.8612		2.54
		0.0526		0.0269		0.0431	
2.55	3.2963		2.2124		2.9043		2.55
		0.0545		0.0278		0.0445	
2.56	3.3508		2.2402		2.9488		2.56
		0.0564		0.0288		0.0461	
2.57	3.4072		2.2690		2.9949		2.57
		0.0585		0.0298		0.0478	
2.58	3.4657		2.2988		3.0427		2.58
		0.0605		0.0309		0.0495	
2.59	3.5262		2.3297		3.0922		2.59
		0.0628		0.0321		0.0513	
2.60	3.5890		2.3618		3.1435		2.60
		0.0652		0.0332		0.0533	
2.61	3.6542		2.3950		3.1968		2.61
		0.0678		0.0345		0.0554	
2.62	3.7220		2.4295		3.2522		2.62
		0.0705		0.0359		0.0575	
2.63	3.7925		2.4654		3.3097		2.63
		0.0734		0.0373		0.0599	
2.64	3.8659		2.5027		3.3696		2.64
		0.0762		0.0388		0.0623	
2.65	3.9421		2.5415		3.4319		2.65
		0.0797		0.0404		0.0650	
2.66	4.0218		2.5819		3.4969		2.66
		0.0829		0.0422		0.0677	
2.67	4.1047		2.6241		3.5646		2.67
		0.0867		0.0439		0.0707	
2.68	4.1914		2.6680		3.6353		2.68
		0.0906		0.0460		0.0739	
2.69	4.2820		2.7140		3.7092		2.69
		0.0946		0.0479		0.0771	
2.70	4.3766		2.7619		3.7863		2.70
		0.0991		0.0502		0.0808	
2.71	4.4757		2.8121		3.8671		2.71
		0.1038		0.0527		0.0846	
2.72	4.5795		2.8648		3.9517		2.72
		0.1090		0.0551		0.0888	
2.73	4.6885		2.9199		4.0405		2.73
		0.1144		0.0579		0.0932	
2.74	4.8029		2.9778		4.1337		2.74
		0.1204		0.0608		0.0980	
2.75	4.9233		3.0386		4.2317		2.75
		0.1266		0.0641		0.1032	
2.76	5.0499		3.1027		4.3349		2.76
		0.1336		0.0675		0.1087	
2.77	5.1835		3.1702		4.4436		2.77
		0.1410		0.0712		0.1148	
2.78	5.3245		3.2414		4.5584		2.78

TABLE 11 : 3

FUNCTIONS FOR THREE-MOMENT EQUATIONS — α, β, AND γ FUNCTIONS FOR AXIAL COMPRESSION — (*Continued*)

L/j	α	$\Delta\alpha$	β	$\Delta\beta$	γ	$\Delta\gamma$	L/j
2.78	5.3245		3.2414		4.5584		2.78
		0.1491		0.0752		0.1213	
2.79	5.4736		3.3166		4.6797		2.79
		0.1579		0.0797		0.1285	
2.80	5.6315		3.3963		4.8082		2.80
		0.1675		0.0844		0.1362	
2.81	5.7990		3.4807		4.9444		2.81
		0.1780		0.0897		0.1448	
2.82	5.9770		3.5704		5.0892		2.82
		0.1894		0.0955		0.1540	
2.83	6.1664		3.6659		5.2432		2.83
		0.2021		0.1017		0.1643	
2.84	6.3685		3.7676		5.4075		2.84
		0.2160		0.1088		0.1757	
2.85	6.5845		3.8764		5.5832		2.85
		0.2315		0.1164		0.1881	
2.86	6.8160		3.9928		5.7713		2.86
		0.2486		0.1251		0.2020	
2.87	7.0646		4.1179		5.9733		2.87
		0.2676		0.1346		0.2174	
2.88	7.3322		4.2525		6.1907		2.88
		0.2890		0.1452		0.2348	
2.89	7.6212		4.3977		6.4255		2.89
		0.3131		0.1573		0.2543	
2.90	7.9343		4.5550		6.6798		2.90
		0.3402		0.1709		0.2763	
2.91	8.2745		4.7259		6.9561		2.91
		0.3710		0.1862		0.3012	
2.92	8.6455		4.9121		7.2573		2.92
		0.4061		0.2039		0.3298	
2.93	9.0516		5.1160		7.5871		2.93
		0.4466		0.2241		0.3625	
2.94	9.4982		5.3401		7.9496		2.94
		0.4933		0.2474		0.4004	
2.95	9.9915		5.5875		8.3500		2.95
		0.5478		0.2747		0.4446	
2.96	10.5393		5.8622		8.7946		2.96
		0.6117		0.3066		0.4964	
2.97	11.1510		6.1688		9.2910		2.97
		0.6876		0.3446		0.5579	
2.98	11.8386		6.5134		9.8489		2.98
		0.7785		0.3901		0.6315	
2.99	12.6171		6.9035		10.4804		2.99
		0.8886		0.4451		0.7209	
3.00	13.5057		7.3486		11.2013		3.00
		1.0238		0.5127		0.8304	
3.01	14.5295		7.8613		12.0317		3.01
		1.1924		0.5970		0.9671	
3.02	15.7219		8.4583		12.9988		3.02
		1.4063		0.7010		1.1405	
3.03	17.1282		9.1623		14.1393		3.03
		1.6834		0.8126		1.3651	
3.04	18.8116		10.0049		15.5044		3.04

TABLE 11 : 3

FUNCTIONS FOR THREE-MOMENT EQUATIONS—α, β, AND γ FUNCTIONS FOR AXIAL COMPRESSION—(*Continued*)

L/j	α	$\Delta\alpha$	β	$\Delta\beta$	γ	$\Delta\gamma$	L/j
3.04	18.8116		10.0049		15.5044		3.04
		2.0513		1.0265		1.6633	
3.05	20.8629		11.0314		17.1677		3.05
		2.5547		1.2782		2.0711	
3.06	23.4176		12.3096		19.2388		3.06
		3.2684		1.6350		2.6498	
3.07	26.6860		13.9446		21.8886		3.07
		4.3300		2.1659		3.5103	
3.08	31.0160		16.1105		25.3989		3.08
		6.0084		3.0051		4.8712	
3.09	37.0244		19.1156		30.2701		3.09
		8.8990		4.4503		7.2138	
3.10	45.9234		23.5659		37.4839		3.10
		14.5332		7.2675		11.7808	
3.11	60.4566		30.8334		49.2647		3.11
		27.9956		13.9987		22.6930	
3.12	88.4522		44.8321		71.9577		3.12
		76.2965		38.1491		61.8440	
3.13	164.7487		82.9812		133.8017		3.13
		1034.4142		517.2088		838.4545	
3.14	1199.1629		600.1900		972.2562		3.14
		∞		∞		∞	
3.15	−227.1668		−112.9747		−183.8716		3.15
		123.4092		61.7055		100.0325	
3.16	−103.7576		−51.2692		−83.8391		3.16
		36.5228		18.2624		29.6049	
3.17	−67.2348		−33.0068		−54.2342		3.17
		17.5035		8.7527		14.1884	
3.18	−49.7313		−24.2541		−40.0458		3.18
		10.2713		5.1365		8.3263	
3.19	−39.4600		−19.1176		−31.7195		3.19
		6.7537		3.3778		5.4750	
3.20	−32.7063		−15.7398		−26.2445		3.20
		4.7787		2.3903		3.8742	
3.21	−27.9276		−13.3495		−22.3703		3.21
		3.5593		1.7807		2.8858	
3.22	−24.3683		−11.5688		−19.4845		3.22
		2.7541		1.3779		2.2330	
3.23	−21.6142		−10.1909		−17.2515		3.23
		2.1940		1.0980		1.7790	
3.24	−19.4202		−9.0929		−15.4725		3.24
		1.7890		0.8954		1.4507	
3.25	−17.6312		−8.1975		−14.0218		3.25
		1.4865		0.7443		1.2057	
3.26	−16.1447		−7.4532		−12.8161		3.26
		1.2548		0.6284		1.0178	
3.27	−14.8899		−6.8248		−11.7983		3.27
		1.0733		0.5376		0.8707	
3.28	−13.8166		−6.2872		−10.9276		3.28
		0.9285		0.4652		0.7533	
3.29	−12.8881		−5.8220		−10.1743		3.29
		0.8111		0.4066		0.6581	
3.30	−12.0770		−5.4154		−9.5162		3.30

TABLE 11 : 3

FUNCTIONS FOR THREE-MOMENT EQUATIONS — α, β, AND γ FUNCTIONS FOR AXIAL COMPRESSION — (*Concluded*)

L/j	α	$\Delta\alpha$	β	$\Delta\beta$	γ	$\Delta\gamma$	L/j
3.30	−12.0770		−5.4154		−9.5162		3.30
		4.6522		2.3367		3.7784	
3.40	−7.4248		−3.0787		−5.7378		3.40
		2.0479		1.0354		1.6681	
3.50	−5.3769		−2.0433		−4.0697		3.50
		1.1477		0.5861		0.9389	
3.60	−4.2292		−1.4572		−3.1308		3.60
		0.7302		0.3785		0.6016	
3.70	−3.4990		−1.0787		−2.5292		3.70
		0.5029		0.2659		0.4179	
3.80	−2.9961		−0.8128		−2.1113		3.80
		0.3647		0.1981		0.3070	
3.90	−2.6314		−0.6147		−1.8043		3.90
		0.2744		0.1544		0.2349	
4.00	−2.3570		−0.4603		−1.5694		4.00
		0.2116		0.1248		0.1854	
4.10	−2.1454		−0.3355		−1.3840		4.10
		0.1662		0.1038		0.1498	
4.20	−1.9792		−0.2317		−1.2342		4.20
		0.1317		0.0887		0.1237	
4.30	−1.8475		−0.1430		−1.1105		4.30
		0.1046		0.0778		0.1036	
4.40	−1.7429		−0.0652		−1.0069		4.40
		0.0826		0.0696		0.0881	
4.50	−1.6603		0.0044		−0.9188		4.50
		0.0641		0.0638		0.0757	
4.60	−1.5962		0.0682		−0.8431		4.60
		0.0810		0.1169		0.1235	
4.80	−1.5152		0.1851		−0.7196		4.80
		0.0238		0.1124		0.0962	
5.00	−1.4914		0.2975		−0.6234		5.00
		0.0568		0.1520		0.0938	
5.25	−1.5482		0.4495		−0.5296		5.25
		0.1964		0.1975		0.0733	
5.5	−1.7446		0.6470		−0.4563		5.5
		0.4898		0.3277		0.0589	
5.75	−2.2344		0.9747		−0.3974		5.75
		1.5111		0.8268		0.0482	
6.0	−3.7455		1.8015		−0.3492		6.0
		25.3412		12.7331		0.0404	
6.25	−29.0867		14.5346		−0.3088		6.25
		∞		∞		0.0048	
2π	$\pm\infty$		$\pm\infty$		−0.3040		2π
		∞		∞		0.0295	
6.5	4.1490		−2.0242		−0.2745		6.5

TABLE 11 : 4

FUNCTIONS FOR THREE-MOMENT EQUATIONS — αh, βh, AND γh FUNCTIONS FOR AXIAL TENSION

L/j	αh	$\Delta\alpha h$	βh	$\Delta\beta h$	γh	$\Delta\gamma h$	L/j
0.00	1.0000		1.0000		1.0000		0.00
		0.0284		0.0163		0.0244	
0.50	0.9716		0.9837		0.9756		0.50
		0.0771		0.0446		0.0664	
1.00	0.8945		0.9391		0.9092		1.00
		0.0097		0.0057		0.0083	
1.05	0.8848		0.9334		0.9009		1.05
		0.0100		0.0058		0.0087	
1.10	0.8748		0.9276		0.8922		1.10
		0.0101		0.0060		0.0089	
1.15	0.8647		0.9216		0.8833		1.15
		0.0105		0.0061		0.0090	
1.20	0.8542		0.9155		0.8743		1.20
		0.0106		0.0062		0.0092	
1.25	0.8436		0.9093		0.8651		1.25
		0.0108		0.0065		0.0094	
1.30	0.8328		0.9028		0.8557		1.30
		0.0110		0.0065		0.0096	
1.35	0.8218		0.8963		0.8461		1.35
		0.0111		0.0066		0.0097	
1.40	0.8107		0.8897		0.8364		1.40
		0.0113		0.0067		0.0098	
1.45	0.7994		0.8830		0.8266		1.45
		0.0113		0.0068		0.0099	
1.50	0.7881		0.8762		0.8167		1.50
		0.0114		0.0068		0.0100	
1.55	0.7767		0.8694		0.8067		1.55
		0.0115		0.0069		0.0100	
1.60	0.7652		0.8625		0.7967		1.60
		0.0115		0.0070		0.0100	
1.65	0.7537		0.8555		0.7867		1.65
		0.0116		0.0070		0.0101	
1.70	0.7421		0.8485		0.7766		1.70
		0.0116		0.0070		0.0102	
1.75	0.7305		0.8415		0.7664		1.75
		0.0116		0.0071		0.0104	
1.80	0.7189		0.8344		0.7560		1.80
		0.0116		0.0071		0.0103	
1.85	0.7073		0.8273		0.7457		1.85
		0.0115		0.0071		0.0102	
1.90	0.6958		0.8202		0.7355		1.90
		0.0115		0.0071		0.0102	
1.95	0.6843		0.8131		0.7253		1.95
		0.0115		0.0071		0.0101	
2.00	0.6728		0.8060		0.7152		2.00
		0.0114		0.0071		0.0101	
2.05	0.6614		0.7989		0.7051		2.05
		0.0113		0.0071		0.0101	
2.10	0.6501		0.7918		0.6950		2.10
		0.0112		0.0071		0.0100	
2.15	0.6389		0.7847		0.6850		2.15
		0.0111		0.0070		0.0100	
2.20	0.6278		0.7777		0.6750		2.20

TABLE 11 : 4

FUNCTIONS FOR THREE-MOMENT EQUATIONS—α_h, β_h, AND γ_h FUNCTIONS FOR AXIAL TENSION — (*Concluded*)

L/j	α_h	$\Delta\alpha_h$	β_h	$\Delta\beta_h$	γ_h	$\Delta\gamma_h$	L/j
2.20	0.6278		0.7777		0.6750		2.20
		0.0111		0.0070		0.0098	
2.25	0.6167		0.7707		0.6652		2.25
		0.0109		0.0070		0.0097	
2.30	0.6058		0.7637		0.6555		2.30
		0.0108		0.0069		0.0098	
2.35	0.5950		0.7568		0.6457		2.35
		0.0107		0.0069		0.0097	
2.40	0.5843		0.7499		0.6360		2.40
		0.0106		0.0069		0.0095	
2.45	0.5737		0.7430		0.6265		2.45
		0.0104		0.0068		0.0095	
2.50	0.5633		0.7362		0.6170		2.50
		0.0103		0.0067		0.0093	
2.55	0.5530		0.7295		0.6077		2.55
		0.0101		0.0067		0.0092	
2.60	0.5429		0.7228		0.5985		2.60
		0.0100		0.0066		0.0092	
2.65	0.5329		0.7162		0.5893		2.65
		0.0099		0.0065		0.0090	
2.70	0.5230		0.7097		0.5803		2.70
		0.0097		0.0065		0.0088	
2.75	0.5133		0.7032		0.5715		2.75
		0.0096		0.0065		0.0088	
2.80	0.5037		0.6967		0.5627		2.80
		0.0094		0.0064		0.0085	
2.85	0.4943		0.6903		0.5542		2.85
		0.0092		0.0063		0.0085	
2.90	0.4851		0.6840		0.5457		2.90
		0.0091		0.0062		0.0085	
2.95	0.4760		0.6778		0.5372		2.95
		0.0090		0.0062		0.0084	
3.00	0.4670		0.6716		0.5288		3.00
		0.0087		0.0061		0.0083	
3.05	0.4583		0.6655		0.5205		3.05
		0.0087		0.0060		0.0080	
3.10	0.4496		0.6595		0.5125		3.10
		0.0085		0.0059		0.0080	
3.15	0.4411		0.6536		0.5045		3.15
		0.0083		0.0060		0.0077	
3.20	0.4328		0.6476		0.4968		3.20

11 : 7. Critical Loads — As the axial load P in a member increases, the value of L/j also increases. When $P = L/j = 0$, there is no axial load to multiply by the deflection and therefore there is no secondary bending, hence the primary bending moment is identical with the total bending moment. For small values of P and L/j the secondary bending is small in comparison with the primary bending but the ratio of the former to the latter increases with an increase in P and L/j. If the axial load is compressive and increases further, a point is reached when

the sum of the series of increments of bending moment constituting the secondary bending becomes infinite. The loading at which this takes place is known as the "critical load," or "critical value of L/j." The critical load and the ultimate load — the load that would cause failure — are not the same except in the case of the ideal Euler strut.

The maximum fiber stress on any cross-section of a member subjected to combined bending and compression is represented by the formula

$$f = \frac{P}{A} + \frac{My}{I}$$

If M_0 is the primary bending moment, and δ the deflection, this formula can be written:

$$f = \frac{P}{A} + \frac{P\delta y}{I} + \frac{M_0 y}{I}$$

As the deflection δ increases, the unit stress f will increase, and the ultimate compressive strength of the material will be reached and the member will fail at a load below the critical. The critical load is therefore not the maximum load that an actual physical member can carry, but rather the load under which the member would be elastically unstable and would fail even if the material had infinite compressive strength.

The expressions for bending moment and deflection are of such character that even though they are used with values of P and L/j greater than the critical, the moments and deflections obtained will be finite. Obviously, however, they do not apply when P and L/j are so large, and it is therefore of importance to the designer to be able to determine the critical load of the member he is designing, or at least be certain that the load to be carried is less than the critical one. This problem has been studied in detail only for the case of a uniformly distributed load, but many of the criteria are equally applicable to other loadings.

Study of any of the formulas for combined bending and compression will show that for any type of lateral load — as the axial load, P, and hence L/j increase, the bending moment and deflection also increase, although the lateral load may remain constant. This increase will continue until the critical value of L/j is reached, when the formulas indicate the bending moment and deflection are infinite. To determine the critical load, however, care must be taken to make sure that the formulas really give the value of infinity for the bending moment or deflection, as there are several special cases where they give the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

The first case for which the critical loading will be studied is that of an axially loaded beam over three supports, with or without overhanging cantilever ends. It will be assumed that the values of L/j for the left-hand span are consistently greater than those in the right-hand span, that the ratio P_1/P_2 is a constant, and that the lateral load on each span is uniformly distributed. As the bending moments M_1 and M_3 over the outer supports are determined by the loads on the cantilever overhangs, if any, they may be treated as known constants, and the moment M_2 over the center support remains to be determined by the three-moment equation. The three-moment equation for this case is Formula 11 : 11 given in Art. 11 : 3. For the present discussion it is advisable to rearrange its terms and write

$$M_2 = \frac{\dfrac{w_1 L_1{}^3 \gamma_1}{4\,I_1} + \dfrac{w_2 L_2{}^3 \gamma_2}{4\,I_2} - \dfrac{M_1 L_1 \alpha_1}{I_1} - \dfrac{M_3 L_2 \alpha_2}{I_2}}{2\left(\dfrac{L_1}{I_1}\beta_1 + \dfrac{L_2}{I_2}\beta_2\right)}$$

As the axial loads in the two spans increase, the coefficients α, β, and γ increase but the formula gives a finite value for M_2. When $L_1/j_1 = \pi$; α_1, β_1, and γ_1 all become infinite and we have the value of M_2 given by the indeterminate expression $\dfrac{\infty}{\infty}$. This expression can be evaluated by the standard method for handling indeterminate forms and then becomes $M_2 = 2\,wj^2 - M_1$, a finite quantity.

If the axial load is further increased, we will have L_1/j_1 greater than π and L_2/j_2 still less than π but both increasing. Study of the tables of α, β, and γ will show that these quantities for the left-hand bay are negative and decreasing in absolute magnitude, while for the right-hand bay they are positive and increasing in magnitude. At first, then, the complex quantity $\dfrac{L_1}{I_1}\beta_1 + \dfrac{L_2}{I_2}\beta_2$ will be negative, the first term being negative and of greater absolute magnitude than the second. As the values of L/j increase further the absolute magnitude of the first term will decrease while that of the second will increase until a point is reached where the whole expression $\dfrac{L_1}{I_1}\beta_1 + \dfrac{L_2}{I_2}\beta_2$ will be equal to zero. When this happens, it is hardly likely that the numerator of the expression for M_2 will also be zero, so we can safely assume that at this loading M_2 becomes infinite, indicating that the loading causing this condition is the critical load.

If the axial loading is still further increased, the absolute magnitude

of $\frac{L_1}{I_1}\beta_1$ will continue to decrease while that of $\frac{L_2}{I_2}\beta_2$ will increase, and their sum will therefore be positive until L_2/j_2 becomes equal to π. When both values of L/j are greater than π, both β_1 and β_2 will be negative.

The criteria for the critical loads for the case being considered can therefore be summarized as follows: If L/j is less than π for both spans the axial loading is less than the critical, but if it is greater than π for both spans the axial loading is greater than the critical. If L/j is less than π for one span and greater than π for the other, and the quantity $\frac{L_1}{I_1}\beta_1 + \frac{L_2}{I_2}\beta_2$ is negative, the axial loading is less than the critical; but if the quantity $\frac{L_1}{I_1}\beta_1 + \frac{L_2}{I_2}\beta_2$ is positive the axial loading is greater than the critical.

The above solution of the problem of determining the criteria for the critical loading in the case considered was first worked out by Professor J. E. Younger of the University of California. Working along similar lines, he also obtained the following criteria for the case of a symmetrically loaded beam over four supports in which $M_1 = M_4$, $P_1 = P_3$, $L_1 = L_3$ and $I_1 = I_3$. If L/j for all spans is less than π, the axial loading is less than the critical, but if L/j is greater than π for all spans the axial loading is greater than the critical. If β_1 and γ_2 are of opposite signs and the algebraic sum $\frac{3\,L_2}{I_2} + \frac{2\,L_1}{I_1}\beta_1 + \frac{L_2}{4\,I_2}\left(\frac{L_2}{j_2}\right)^2\gamma_2$ is negative, the axial loading is less than the critical; whereas if that sum is positive, the axial loading is greater than the critical. In this case also when $L_1/j_1 = \pi$, $M_2 = 2\,wj^2 - M_1$. When $L_2/j_2 = \pi$, $M_2 = M_3 = wj^2$ and $M_3 = 2\,wj^2 - M_2$.

The criteria listed above are good only for the two special cases of continuous beams where the lateral load is uniformly distributed over each span and there are only two spans (case 1) or where there are three spans with both the beam and its loading symmetrical (case 2). The criterion that when L/j is less than π for all spans, the critical loading has not been reached is believed to apply to all types of loading and any number of spans. Similarly the criterion that if L/j is greater than π in all spans the critical loading has been exceeded is believed to be general except for one apparent exception. This exception is the case of two symmetrically loaded spans, for which the critical loading is that giving $L/j = 4.49+$ in both spans. For practical purposes, however, this exception is apparent rather than real, as the slightest deviation from symmetry of either the construction of the

beam or the loading on it would result in failure whenever L/j is greater than π.

The only debatable cases are those in which L/j for some of the spans is less than π while for the others it is greater. The criteria for the two most important of these cases have been listed above, and those for any other case could be worked out by the same method, though the practical advantages of doing so might not justify the labor involved. In most practical designs, when L/j in any span is greater than π, the deflections and secondary bending moments are so great that the beam would fail due to excessive compression on its most stressed fibers. It is therefore a good rule to avoid all designs involving values of L/j greater than π.

The discussion above applies only to the bending moments over the supports of a continuous beam, and it is obviously necessary to investigate the bending moments between the supports also.

The formula for the bending moment at any point in a single span subjected to axial compression and a uniformly distributed lateral load is

$$M = \frac{D_2 - D_1 \cos \frac{L}{j}}{\sin \frac{L}{j}} \sin \frac{x}{j} + D_1 \cos \frac{x}{j} + wj^2$$

This expression for M is determinate for all values of L/j less than π; but when $L/j = \pi$ then $\sin L/j = 0$ and the first term appears to become infinite. In most cases it will become infinite, indicating that the critical load is such that $L \div \sqrt{\frac{EI}{P}} = \pi$ or $P = \frac{\pi^2 EI}{L^2}$. This value of P is the Euler load for the member if considered as a pin-ended column, and its appearance as the critical load for this case shows that the latter is independent of the lateral load on the member.

There is one case, however, in which π is not the critical value for L/j in the above formula. If $D_2 = D_1 \cos \pi$, the expression for M takes the indeterminate form $\frac{0}{0} + D_1 \cos \pi + wj^2$. As $\cos \pi = -1$, we would get this result if $M_2 - wj^2 + M_1 - wj^2 = 0$ or $M_2 = 2\,wj^2 - M_1$. As shown above, however, this is just the value obtained for M_2 when $L/j = \pi$ in a continuous beam in both of the cases investigated. It is probable that a similar result would be obtained for a continuous beam subjected to any other type of lateral load.

If the span being investigated is the only one, π is the critical value for L/j. If it is one of a series, it is not the critical load but the critical load for the beam is that which will cause an infinite moment over

one of the supports. Such a result may seem unreasonable at first glance but a little study will show that it is quite in line with what should be expected.

If the axial compression in the span shown in Fig. 11 : 12 is such that $L/j = \pi$, the center portion of the span will tend to deflect infinitely — and will deflect, in fact, until the material fails. The loads on the cantilevers may give the beam a little reversed curvature at the supports, but they will not be able to prevent rotation of the beam about those supports due to the action of the combined lateral and axial loads in the span. In the case of a continuous beam, the conditions are different. Suppose $L/j = \pi$ in the left-hand span of the beam in Fig. 11 : 13 and is less than π in the right-hand span. While the load on the left-hand cantilever cannot prevent rotation of the beam about support 1, rotation about support 2 is limited by the action of the right-hand span and the reaction at support 3. The bending moment at support 2 depends in part upon this action; and when M_2 is such that

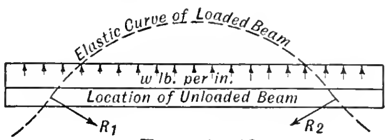


FIG. 11 : 12

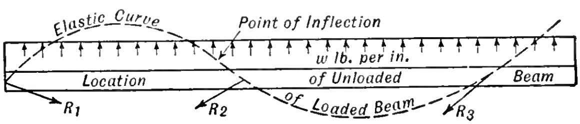


FIG. 11 : 13

$D_1 = D_2 \cos \pi$, a condition of equilibrium is reached. The result is that there is a point of inflection fixed a short distance to the left of support 2 and the span may be considered as made up of two parts: a cantilever from support 2 to the point of inflection, and a beam with one end restrained and the other pinned between support 1 and the point of inflection. The latter beam will have a value of L/j less than π and will thus be determinate.

11 : 8. Special Computation Devices — In addition to those already mentioned in the discussion of critical loads, there are certain cases where the formulas are difficult to use either because they give the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or the quantities to be taken from the tables are varying with such rapidity that straight line interpolation is of insufficient precision. For the more common of these cases there are special methods of computation with which the student should be familiar.

In the special case when $M_1 = wj^2$, $D_1 = 0$. Substituting in the formula for the location of the point of maximum moment we obtain $\tan x/j = D_2/0 = \infty$, whence $x/j = \pi/2$ and $\cos x/j = 0$. Substituting these values in the formula for the maximum bending moment in the span

$$M_{\max} = \frac{D_1}{\cos x/j} + wj^2 = \frac{0}{0} + wj^2$$

which is an indeterminate expression. This difficulty can be avoided by using the expression for bending moment at any point with $\pi/2$ for x/j. That expression then becomes

$$M = \frac{D_2}{\sin L/j} + wj^2$$

When D_1 is small, though not zero, $\tan x/j$ varies rapidly, and it is impossible to obtain values of x/j and $\cos x/j$ with sufficient precision by the ordinary method of interpolation. The simplest method of attack in such a case is to interchange D_1, and D_2 in the formulas for $M_{\max}$. Then x will be measured from the right-hand support; that is, the interchange of D_1 and D_2 is equivalent to turning the beam end for end. Unless $\frac{L}{j}$ is very close to π in value, this method will alter the value of $\frac{x}{j}$ sufficiently to permit a solution without recourse to any special formulas.

A second method depends on the fact that

$$\frac{1}{\cos A} = \sec A = \sqrt{\tan^2 A + 1}$$

Applying this relationship to the expression for maximum moment in the span and substituting for $\tan\frac{x}{j}$ its value in terms of D_1 and D_2,

$$M_{\max} = \frac{1}{\sin\frac{L}{j}}\sqrt{D_1^2 + D_2^2 - 2\,D_1\,D_2\cos\frac{L}{j}} + wj^2$$

By using this formula it is possible to find the maximum moment without solving for $\frac{x}{j}$, but the location of the moment is, of course, not obtained.

A third method, which is slightly approximate, depends on the fact

that when $\frac{x}{j}$ is near $\frac{\pi}{2}$, $\sin\frac{x}{j}$ is nearly equal to 1, so that $\frac{1}{\cos\frac{x}{j}}$ may be equated to $\tan\frac{x}{j}$. Then,

$$M_{\max} = D_1 \tan\frac{x}{j} + wj^2 = \frac{D_2 - D_1\cos\frac{L}{j}}{\sin\frac{L}{j}} + wj^2$$

This last method involves an error of less than 1 per cent in the first term of the equation when $\frac{x}{j}$ is between 1.45 and 1.70.

11 : 9. Approximate Methods — There are several approximate formulas and methods that are often useful, particularly in preliminary design. Thus if the first few terms of the infinite series are substituted for the secant term in the formula for maximum moment on a pin-ended strut with uniformly distributed lateral load (Case B′, Art. 11 : 5), it becomes

$$M = \frac{M_0}{1 - \frac{5\,PL^2}{48\,EI}}$$

where $M_0 = -wL^2/8$ the primary bending moment due to side load alone. If for 5/48 we substitute 1/10 we obtain Johnson's approximate formula for this case, while the substitution of $1/\pi^2$ will give Perry's formula,

$$M = M_0\frac{Q}{Q - P}$$

where Q is the Euler load of the strut in question.

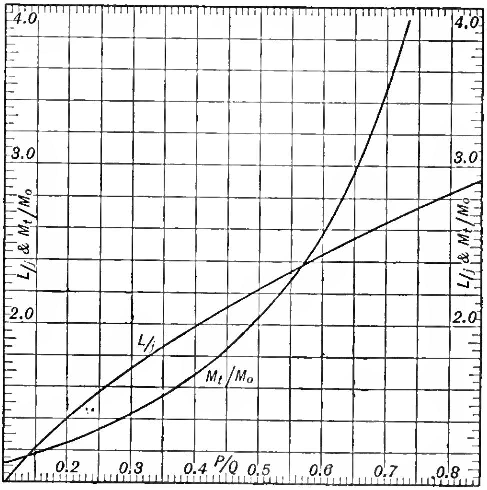


FIG. 11 : 14

The formulas above show that for a pin-ended strut with uniformly distributed lateral load there are definite relationships between the ratios M/M_0, L/j, and P/Q. In Fig. 11 : 14 are plotted curves showing the values of the two former ratios in terms of P/Q. These curves can be used to compute the maximum total bending moment on this type

of strut, and are also very useful for approximate computations of the total moment under other loadings.

11 : 10. Effect of Change in Axial Load Between Supports — The drag truss bays in an airplane are generally shorter than those of the lift truss, there usually being two or three drag bays to one lift bay. The stresses in the drag, or anti-drag, wires produce axial loads in the spars so that the value of P cannot be taken as constant throughout a lift truss bay. In addition to this variation in the axial load the spar sections themselves are often varied in the different drag bays so that the moment of inertia is not constant. The precise formulas developed in the foregoing articles are therefore not accurate for these conditions since they assume a constant axial load and a constant moment of inertia between supports. The following method may be used for approximating the moments on the spars under such circumstances.

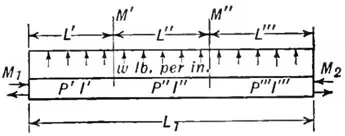


FIG. 11 : 15

Figure 11 : 15 shows one bay of a lift truss of length L_1, which is subdivided by the drag truss into three bays of length L', L'' and L'''. The axial loads are P', P'', and P''', and it will be assumed that the section is changed in each bay so that the moments of inertia are I', I'', and I''' respectively.

$$\text{Let } P_1 = \frac{P'L' + P''L'' + P'''L'''}{L' + L'' + L'''}$$

$$I_1 = \frac{I'L' + I''L'' + I'''L'''}{L' + L'' + L'''}$$

and compute j in the usual way. This value of j should be used in the three-moment equation.

This method of using weighted values for P and I is somewhat more accurate than that of using the arithmetical mean but the difference is so slight as to be negligible in most cases. Where the changes in load or section between bays is great this method of weighting the values of P and I should be used, but it is, in most cases, an unnecessary refinement.

11 : 11. Secondary Shear — In most beam computations the shear at any point along the beam axis is assumed to be the shear on a section through that point and perpendicular to the location of the beam axis before the loads were applied. For purposes of web design it would be more correct to compute the shear on sections perpendicular to tangents to the elastic curve of the beam at the points in question. Since the angle between such a tangent and the original location of the

beam axis (the slope i) is always assumed to be very small, the usual practice is justified; first, because the error is both negligible and on the safe side; and second, because the computations are greatly simplified.

In the cases of beams subjected to axial loads that are large in comparison to the transverse loads (and most airplane spars fall into this class), the error involved in the usual method of computing the shear is no longer negligible, and for proper design the shear on sections perpendicular to the elastic curve must be computed. In a study of airplane spar tests made to determine the constants to be used in the design of webs for box beams, Roy A. Miller[1] found cases in which the shear on sections perpendicular to the elastic curve was as much as 20 per cent greater than that on sections perpendicular to the original location of the beam axis. His results also showed that this additional shear should be taken into account in design.

The magnitude of the total shear on a section perpendicular to the elastic curve of a beam can easily be computed. In Fig. 11 : 16, CF is the elastic curve of the beam; CD, a tangent to CF at C; DE, the original location of the beam axis before loading; and AB, a section through the beam at C and perpendicular to DC. In the same diagram the forces acting on the section AB from the portion to the left are represented by P and S_0, parallel and perpendicular, respectively, to the original axis DE. P is what is usually thought of as the axial load and S_0, the shear as usually computed.

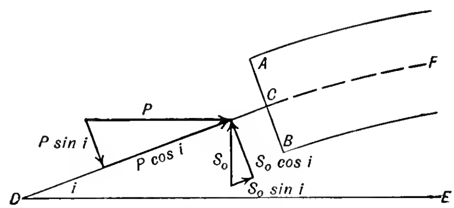


Fig. 11 : 16

The normal force on section AB is then obviously equal to $P \cos i + S_0 \sin i$ and the shear force is $S_0 \cos i - P \sin i$. If the slope, i, is assumed to be small, $\cos i = 1.00$ and $\sin i = i$, practically. Then the normal force on section AB will be $P' = P + S_0 i$, and the shear will be $S = S_0 - Pi$.

From the derivation of the formulas for bending moment in Art. 11 : 2 it should be obvious that

$$M = M_0 - P\delta \tag{11 : 13}$$

in which M is the total bending moment; M_0 is the bending moment due to transverse loads alone; P is the load parallel to the original axis of the beam, positive when in compression, and assumed to be constant; and δ is the deflection (positive when upward).

[1] Air Corps Information Circular 516 "Design of Plywood Webs for Box Beams."

Differentiating equation 11 : 13 with respect to x,

$$\frac{dM}{dx} = \frac{dM_0}{dx} - P\frac{d\delta}{dx} \qquad 11:14$$

From Art. 3 : 8, Chapter III, however, $\frac{dM_0}{dx} = S_0$, the shear on a section perpendicular to the original axis of the beam, and $\frac{d\delta}{dx} = i$, the slope of the elastic curve. Hence equation 11 : 14 may be written,

$$\frac{dM}{dx} = S_0 - Pi \qquad 11:15$$

or the derivative of the total bending moment with respect to x is equal to the shear on a section perpendicular to the elastic curve of the beam. Thus $S = \frac{dM}{dx}$ is seen to express the relationship between the most important shears and the total bending moment on a beam subjected to axial compression.

If the expression for the true shear S is differentiated with respect to x, we have $\frac{dS}{dx} = \frac{dS_0}{dx} - P\frac{di}{dx}$, the axial load P being assumed constant. The first term on the right side of this expression is the rate of change of load normal to the original axis, and the second term the rate of change of the component of P perpendicular to the elastic curve. Thus we can also say that $S = \int w\,dx$ for the axially loaded beam as well as for that without axial load. When using this general differential equation for the case of an axially loaded beam, however, it must be remembered that the loads and shears represented are those perpendicular to the elastic curve, and that parts of them are components of the axial load P parallel to the original axis of the beam. The objection may be made to the above illustrations of the applicability of the general differential equation for beams to the relations between load, shear, and bending moment, that the bending moment at the section has not been considered. Such an objection is not valid, however, as the only effect of the bending moment is to change the point of application of the load P. It would not affect the equations of equilibrium on which the illustrations are based.

In the derivation of the equations given in the tables in this chapter, the only deflections considered were those due to bending. In addition to such deflection, there is a further deflection due to shear. In Air Corps Information Circular 493, "The Investigation of Structural

Members under Combined Axial and Transverse Loads — Section II," the authors have investigated this phase of the problem and found that the result of this shear deflection is to reduce the critical load below that indicated by the formulas. A similar problem arises in the cases of beams without axial load, as the total deflection is the sum of that due to bending and that due to shear. The usual procedure for both classes of beams is to use a standard value for the modulus of elasticity somewhat smaller than the true value. Using this standard value of E in the computations results in larger computed deflections than would be obtained from the use of the true value, the difference being an allowance for shear deflection. Though this is a rule-of-thumb method and is obviously inaccurate, it gives satisfactory results and is almost universally used. It is advisable to make precise computations of the shear deflections only in the interpretation of the results of very precise tests. The unavoidable differences between the standard value of E used in design and the actual E of the material that gets into the structure, and the factor of safety used to cover these differences, are quite sufficient to offset the error involved in the neglect of the shear deflection in practical design.

11 : 12. Notes on the Use of the Precise Formulas — It is recommended that at least four significant figures be used in all computations involving the formulas in this chapter. In preliminary investigations or for the purpose of obtaining a rough check on a spar, three figures will be sufficient but, since the final result of several of these formulas depends on small differences between large quantities, three significant figures will often give misleading results. This fact should be borne in mind and the number of significant figures necessary to give the required precision in the results should be used. This matter is especially important when the value of L/j is near π, as the functions α, β and γ are all changing rapidly in that range.

Special care must be taken to use the proper signs throughout the computations or serious errors will result. This is particularly true in the case of the signs of the terms for loads and deflections. The conventions for signs are given in Art. 11 : 5 in this chapter.

In applying the precise formulas to design, it is necessary to decide upon a size of member before the final values of the bending moments can be obtained. This makes the process of design a matter of successive trials; but by first computing the bending moments and axial loads without allowing for secondary stresses and using those moments and loads suitably modified by the judgment of the designer on the first trial, the number of trials needed to obtain a satisfactory design should not be excessive.

In discussions of the precise methods of computing stresses due to combined loadings in this volume and by other authors (as Cowley and Levy in "Aeronautics in Theory and Experiment," etc.), failure is usually assumed to mean failure due to elastic instability or "buckling." Usually, before such failure would occur in practice, the member would have failed by rupture of the material due to excessive unit stresses, and statements regarding the criteria for failure must be read with these facts in mind. The criteria for failure in buckling implicitly assume that the material has a constant finite modulus of elasticity and an infinite proportional limit and ultimate allowable stress. Of course, no engineering material has such properties, but the precise formulas and resulting criteria for buckling failure are nevertheless very useful in determining the loads under which failure by rupture of the material is likely to occur.

11 : 13. Graphical Methods — It will be noted that all the analytical expressions for combined axial and transverse loads developed thus far have been based on the assumption that the moment of inertia of the beam was constant throughout each bay. The assumption that I varies leads to a differential equation that is so cumbersome as to appear incapable of solution, at least in a form satisfactory for practical use. As a matter of fact, the equation has not been solved, except for one or two special cases, one of which involves the use of Bessel's functions. For this reason precise formulas that are applicable to tapered beams such as those used in many of the smaller airplanes, have not yet been derived. In some cases, satisfactory results may be obtained by using, in the formulas based on a constant I, a mean value of the moment of inertia in each bay. Such a method is inherently approximate and is not to be recommended for the design of small wing-spars in which the margin of safety is likely to be low.

A graphical method has been devised by Miss Barbara Gough.[1] It is applicable to continuous beams with three supports, the bending moments at two of which are known, for any type of transverse load and any variation in cross-section. The axial load is assumed to be compressive.

This method, while slightly approximate and subject to the errors of any graphical method, is simple in its application and gives quite accurate results. It has been found to be capable of extension to cover the case of a continuous beam with four supports if the structure and loading are symmetrical so that the moments at the two central supports are identical. As this is the condition existing in almost all

[1] R. and M. 741, "A Graphical Method for the Determination of Bending Moment and Deflection of Aeroplane Spars."

single-bay, tapered-wing airplanes, the graphical method has a considerable field of application.

Fig. 11 : 17 shows a complete solution of the method as applied to the wing-spar of a small airplane.

Procedure for Span 1–2 (Fig. 11 : 17 (c)): — First divide the span into n equal sections — five or six will generally suffice from the standpoint of accuracy. Then choose the ratio of reduction in horizontal

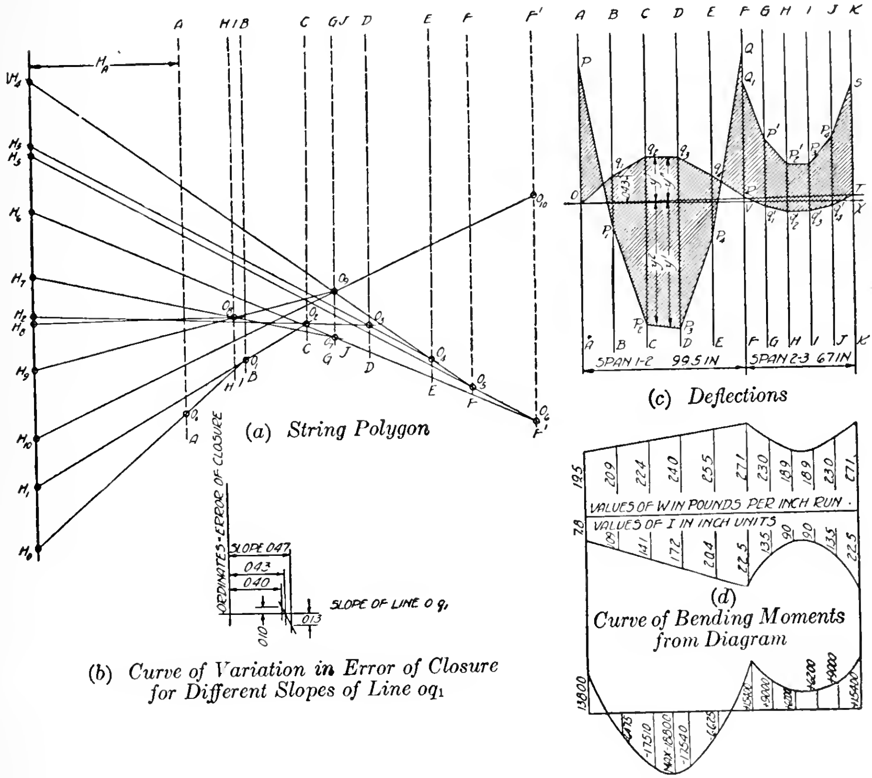


FIG. 11 : 17

scale, λ, necessary to keep the solution on the sheet of paper to be used. With these two values decided upon, compute the polar distances, H_A, H_B, H_C, H_D, H_E, and H_F (Fig. 11 : 17 (a)), from the formula,

$$H = \frac{EIn}{P\lambda L}$$

in which E = the modulus of elasticity.

I = the moment of inertia of the member at the section being investigated.

n = the number of sections (5 in Fig. 11 : 17 (c)).
λ = the scale of reduction (40 in Fig. 11 : 17 (c)).
L = the actual, full-sized length of the span.

The computations for these values will be found in Table 11 : 5.

Attention is called to the fact that the polar distances vary inversely with λ and that if the scale of reduction is small the polar distances will be so large that the intersection of the rays will not fall on the paper. These polar distances should be plotted from a common base line, drawn at right angles to the OX-axis of the beam, and the lines AA, BB, etc., drawn in (see Fig. 11 : 17 (a)).

The values of y' for Span 1–2 (Fig. 11 : 17 (c)), are next computed. For the type of loading used y' is found from

$$y' = \frac{M_1}{P} + \frac{Vx}{P} + \frac{wx^2}{2\,P} + \frac{Kwx^3}{6\,LP}$$

in which V represents the shear just to the right of Support 1; and M_1 is the bending moment at Support 1 and must be a known quantity, such as that from a cantilever wing tip — it amounts to 13,800 in.-lb., in this example. The last term, $\frac{Kwx^3}{6\,LP}$, is the primary bending due to the triangular segment in the loading curve representing the increment in load between two consecutive sections divided by P. V, however, depends on the moment at the second support, which is one of the things to be found and so is unknown. It is necessary, therefore, to assume a value for V, the most logical being that found by the use of the ordinary three-moment equation. In the illustrative example, the value used for V was obtained from an approximate analysis of the beam, but the exact value chosen has little influence, because the method of construction eliminates the effects of small differences in V. In computing the values of y', x is the actual full-scale length of the member, not the length of the reduced scale diagram (Fig. 11 : 17).

The values of y' are then plotted as ordinates from the horizontal axis OX, giving the points, p, p_1, p_2, etc. The next step is to assume a slope for, and draw in, the line, Oq_1, and the string, H_0O_0, parallel to it. The location of the point H_0 is chosen arbitrarily to bring the completed polar diagram on the paper. Ray H_0O_0 is continued until it intersects the axis, BB, at O_1. The distance p_1q_1 is now scaled from the string polygon and laid off from H_0 along the base line to locate H_1. H_1O_1 is then drawn in and continued to O_2 on the axis, CC. q_1q_2 is next drawn parallel to this line and the distance, p_2q_2, scaled and laid off from H_1 to locate H_2. This procedure is continued until the point, R (Fig. 11 : 17 (c)), is reached on the line through the intermediate support.

TABLE 11 : 5

Computation of Polar Distances H, and Values of the Deflections y'

	Span 1–2. $P_1 = 6750$; $n = 5$; $\lambda = 49$; $L = 99.5$						Span 2–3. $P_2 = 8750$; $n = 5$; $\lambda = 40$; $L = 67.0$					
	A	B	C	D	E	F	F′	G	H	I	J	K
I	7.8	10.9	14.1	17.2	29.4	22.5	22.5	13.5	9.0	9.0	13.5	22.5
$\frac{E}{P}$	237	237	237	237	237	237	183	183	183	183	183	183
$\frac{EI}{P}$	1837	2586	3334	4084	4832	5333	4118	2471	1647	1647	2471	4118
$\frac{n}{\lambda L}$	0.00125	0.00125	0.00125	0.00125	0.00125	0.00125	0.00187	0.00187	0.00187	0.00187	0.00187	0.00187
H	2.31	3.25	4.19	5.13	6.07	6.70	7.66	4.61	3.07	3.07	4.61	7.66
$\frac{M_1}{P_1}$	+2.04	+2.04	+2.04	+2.04	+2.04	+2.04						
$\frac{V}{P_1}$	−0.159	−0.159	−0.159	−0.159	−0.159	−0.159						
x	0	19.9	39.8	59.7	79.6	99.5						
$\frac{Vx}{P}$	0	−3.16	−6.32	−9.49	−12.66	−15.82						
$\frac{W}{2P}$	0.00143	0.00143	0.00143	0.00143	0.00143	0.00143						
x^2	0	396	1584	3564	6336	9900						
$\frac{Wx^2}{2P}$	0	+0.57	+2.27	+5.10	+9.06	+14.16						
$\frac{Kw}{6LP}$	0	0.00000194	0.00000194	0.00000194	0.00000194	0.00000194						
x^3	0	7880	63045	212776	504358	985075						
$\frac{Kwx^3}{6LP}$	0	+0.02	+0.12	+0.41	+0.98	+1.91						
y'	+2.04	−0.53	−1.89	−1.94	−0.58	+2.29	0	−0.86	−1.25	−1.25	−0.86	−0

Values of y' for span 2–3 are computed for $y' = \frac{M}{P_2}$, where M is the moment due to the transverse load only, assuming the span to be a simply supported beam.

Assumed value of $V = -1075$ lb. E for spruce taken as 1,600,000 in.-lb. (obselete standard based upon 10% moisture).

In span 2–3: $M_G = M_J = 7538$; $y'_G = y'_J = 0.86$. $M_H = M_I = 10940$; $y'_H = y'_I = 1.25$. $RQ_1 = \frac{P_1}{P_2} RQ = \frac{6750}{8750} 2.25 = 1.74$.

RQ (Fig. 11 : 17(*c*)) is a measure of the bending moment at this point and, as continuity is required in the two spans, it is necessary to multiply the ordinate by a factor such that the value obtained from the product of RQ and the axial load in Span 1–2 will be the same as the product, RQ_1, times the axial load in Span 2–3. This factor is obviously $\frac{P_1}{P_2}$, so that $RQ_1 = \frac{P_1}{P_2} RQ$.

It may be shown that the ordinates, y'', to the curve, $Oq_1q_2q_3q_4R$ (Fig. 11 : 17(*c*)), measured from the axis OR, are equal to the deflections in Span 1–2. Hence, if all three supports lie on the same straight line, all the deflections at O, R, and T must be zero and a straight line through O establishes the point, T.

The problem now resolves itself into constructing the curve, $Rq_1'q_2'q_3'q_4'T$, so that it will satisfy the conditions in Span 2–3 and close on the point, T, with zero deflection.

Procedure for Span 2–3:—Divide Span 2–3 (Fig. 11 : 17(*c*)) into n equal sections, not necessarily the same number as was used in Span 1–2, and compute the polar distances H_F, H_G, H_H, H_I, H_J and H_K in the same way as in Span 1–2. Draw in the lines, $F'F'$, GG, etc., through the poles (Fig. 11 : 17(*a*)). The values of n, I, P, and L for Span 2–3 should, of course, be used.

If the bending moment at the third support is zero, as in the case of a pin-joint, the ordinate, ST, must be zero, because this ordinate, when multiplied by P_2, gives the value of the bending moment. For such a condition the line, Q_1S, (Fig. 11 : 17(*c*)), would become Q_1T, and the curve, $Q_1p_1'p_2'p_3'p_4'S$, would close on T. However, in the illustrative example the beam and loading are symmetrical about the mid-point of Span 2–3, so that $M_2 = M_3$, hence, the ordinate, TS, $= RQ_1$. The curve, $Q_1p_1'p_2'p_3'p_4'S$ (Fig. 11 : 17(*c*)), is constructed by plotting the ordinates, y', from the line Q_1S. The values of y' are obtained from $y' = \frac{M}{P_2}$, in which, M is the moment at the section due to transverse load only, assuming the moments at Supports 2 and 3 to be zero.

The curve, $Q_1q_1'q_2'q_3'q_4'T$ (Fig. 11 : 17(*c*)), is then constructed. It is to be noted (Fig. 11 : 17(*a*)) that there are two computed polar distances for F, one based on the axial load in Span 1–2, and the other on that in Span 2–3. The slope of the line, Rq_1' is obtained by laying off one-half of the ordinate, RQ, from H_4 to obtain H_5. Line H_5O_5 is drawn and continued to O_6 on the axis, $F'F'$ (Fig. 11 : 17(*a*)). One-half of RQ_1 is then laid off from H_5 to locate H_6 and the line H_6O_6 is drawn in. Rq_1' is next drawn parallel to H_6O_6. H_7 is determined as in the case of Span 1–2 by making H_6H_7 equal to the ordinate, $p_1'q_1'$,

and the slope of $q_1'q_2'$ is the same as the ray, H_7O_7, O_7 being located at the point of intersection of the ray, H_6O_6, with the axis, GG. The construction is continued in this way and $q_4'T$ should close on T. It is obvious that this will not occur unless the correct slope has been assumed for the line, Oq_1. By measuring the error of closure and plotting it against the assumed slope, as is done in Fig. 11 : 17(*b*), the trend of the error of closure may be found by drawing a curve through the plotted points. This curve will indicate the slope necessary for zero error of closure so that the third trial should come very near to closing and the fourth close exactly if the drafting is done carefully in each case.

In some cases the H–H distances are laid off above the last point; in others, below it. They are to be drawn in the same direction as the pq intercept at the section, that is, above when q is above p, and below when q is below p.

If the loading is not uniformly varying in the span, the equation for Py' must obviously be altered to give the moment at any point for the type of loading used. If the points of support do not lie on a straight line this can be provided for by having the $q_1q_2q_3$-curve intersect the lines at the supports at a distance above, or below, O, R and T, equal to the known deflections of these points.

When the diagram has been completed and gives a satisfactory closure on T, the bending moments at any point may be found by scaling the vertical distance between the $p_1p_2p_3$ and $q_1q_2q_3$-curves (Fig. 11 : 17(*c*)) and multiplying by the axial load at the point. If the deflection at any point is wanted, the vertical intercept between the $q_1q_2q_3$-curves and the line, ORT, gives the actual value.

Fig. 11 : 17 gives all the data pertaining to the rear upper beam of a small airplane and, with Table 11 : 5, gives all the work required for the complete determination of the bending moments and deflections of this beam, except that the diagrams obtained from the first trials are not shown. The cross-hatched area represents the area between the curves used in deriving the bending moments at any point, the depth of this area, in inches, times the axial load, in pounds, giving the moment in inch-pounds.

This method gives the bending moments at all points on the span, which is a very desirable feature when designing a tapered beam, since the critical section may be at a point other than the section of maximum bending moment if the rates of change of bending moment and of the section modulus of the beam are not the same.

PROBLEMS

11 : 1. Substitute a spar having the cross-sectional dimensions shown in Fig. 10 : 7 for that shown in Fig. 11 : 4 and draw bending moment curves for the spans and loadings used in Art. 11 : 4.

11 : 2. A 2″–0.065 chrome-molybdenum steel tube is tested under combined axial and lateral loads. It is 60 in. long and the ends bear on knife-edges which are centered so that they introduce no moments at the ends of the tube. The axial load, which produces compression in the tube, is 5000 lb. The lateral load of 500 lb. is applied at mid-span. What is the maximum fiber stress in the tube?

11 : 3. Draw shear curves for the spar shown in Fig. 11 : 4 under the loadings used in Art. 11 : 4 first neglecting the effect of the axial load and then taking it into account. What is the maximum difference between the values of the shears at either support expressed in per cent of the values obtained when the axial load is neglected?

CHAPTER XII

ALLOWABLE STRESSES IN COMBINED BENDING AND COMPRESSION

Authoritative and well correlated data on the allowable strength properties of materials subjected to combined bending and compression are, unfortunately, scarce. While satisfactory data are available as to strength of spruce in various shapes and of aluminum alloy in tubular form, practically no information is available on any of the various grades of steel in any form, or on aluminum alloy in shapes other than tubes. Since the effect of shape is of great importance, what few data are available on one type of section cannot be extended to cover others and recourse must be had to tests to prove the strength of a given design.

12 : 1. Form Factors and Shape Effects — The effect of shape is most pronounced in wooden members such as the I and box sections used for wing-spars. A short strut of aircraft spruce having a moisture content of 15 per cent will, if subjected to an axial load parallel to the grain, fail by crushing when the intensity of stress at any cross-section reaches approximately 5000 lb. per sq. in. A square specimen of the same material, if supported as a simple beam and subjected to pure flexure, will fail when the intensity of stress on the extreme fiber as indicated by $f = \dfrac{My}{I}$ reaches approximately 9400 lb. per sq. in. This latter value, the modulus of rupture, is very nearly twice the maximum stress obtainable in simple crushing parallel to the grain. A question which immediately presents itself is, "At what stress intensity will failure occur on a member subjected to both axial loads and flexure?" Furthermore, "What will happen if the member is in the long column class so that its resistance to axial loading is low?" and "What will happen if it is not of square cross-section?" Let us consider the last of these questions first.

The United States Forest Products Laboratory at Madison, Wisconsin, has found[1] that the strength properties of spruce of a quality suitable for aircraft use average about as follows:

[1] Reports 181 and 188 of the National Advisory Committee for Aeronautics. "The Influence of the Form of a Wooden Beam on its Stiffness and Strength," by J. A. Newlin and G. W. Trayer.

Moisture Content	15 per cent.
Modulus of Elasticity	1,300,000 lb. per sq. in.
Modulus of Rupture	9,400 lb. per sq. in.
Stress at Elastic Limit in Bending	6,200 lb. per sq. in.
Maximum Compression Parallel to the Grain	5,000 lb. per sq. in.
Stress at Elastic Limit in Compression	4,000 lb. per sq. in.

The modulus of rupture and stress at elastic limit in bending were obtained for standard specimens 28 inches long and 2 by 2 inches in cross-section. Specimens of other shapes were found to have different properties. For example, a circular stick having a cross-sectional area of 4 sq. in. was found to be capable of supporting practically the same load, as a beam, as the standard specimen although the section modulus, $\frac{I}{y}$, for a square is about 18 per cent greater than that for the circle. Moreover, a two-inch square section tested with one of its diagonals vertical will carry the same bending moment as the standard section tested with its sides vertical and horizontal although the section modulus in the latter position is about 41 per cent greater. Hence, if M be the same and $\frac{I}{y}$ be different the apparent fiber stresses at failure, the moduli of rupture, must vary for these different shapes. The results of many tests indicate that such a variation actually exists.

The explanation for this lies in the fact that for materials such as spruce, stress does not vary directly with strain, hence, if plane cross-sections before bending remain plane after bending, as they apparently do, the strain in any particular fiber will depend directly on its distance from the neutral axis, but the stress will not. As a consequence the neutral axis will not pass through the centroid of the cross-section but will shift toward one extreme fiber or the other. The ordinary beam formula, $f = \frac{My}{I}$, is, therefore, no longer applicable. Bach[1] discusses this effect quite fully and develops equations for determining the stresses in beams which, while theoretically correct, are much too cumbersome for use in practical design.

The following concept, which was developed by the Forest Products Laboratory, gives a clear understanding of the importance of shape effect. It assumes that the beam theory does hold and that, within the elastic limit, stresses are proportional to the distances from the neutral axis of the fibers in which they occur. The ability of the com-

[1] Elastizitat und Festigkeit," by C. Bach, 8th edition, pages 286 *et. seq.*

pression fibers in a beam to carry a higher stress than those in a strut may be accounted for by considering each of the minute wood fibers to act as an Euler column. In the strut each such fiber is stressed to the same degree as those around it so that none can offer much support to its neighbors. Hence, when one fiber yields it throws additional loads on those surrounding it and they all yield, assuming all to have about the same strength. The failure may thus be accounted for as a progressive failure of the fibers all of which are carrying about the same load so that none can help the others. In the case of a beam, however, all of the fibers are not equally stressed so that the less stressed fibers can support those carrying the maximum loads, or those which have been stressed slightly beyond their elastic limit. Assuming that the less stressed fibers will support the more heavily stressed by carrying part of the load when the heavily stressed fibers pass their elastic limit and yield slightly, the failure of the beam would be a progressive failure of its fibers and would not occur until what appears to be a very high stress had been developed in the extreme fibers.

This effect can best be illustrated, perhaps, by a consideration of a square beam tested with one diagonal vertical. The most stressed fiber may then be visualized as a single element extending the length of the upper corner of the beam, which is the corner carrying the maximum compressive stress when the load acts vertically downward. If, then, it is assumed that there are two elements in the layer next below the corner, each will be subjected to a stress less than that in the extreme fiber and, if the extreme fiber passes its elastic limit and yields slightly, these adjacent fibers will immediately absorb the slight increment of load. Similarly, when these two elements reach their elastic limits, the three or four in the next layer below them will absorb the extra stresses due to their yielding, and this action will continue through successive layers so that several increments may be added before the beam as a whole will appear to have passed its elastic limit. This same conception of the absorption of stress from a fiber that has yielded may be extended to one that has failed, and it is immediately apparent that the number of fibers adjacent to and supporting the critical one is of the utmost importance. This, of course, depends on the shape of the cross-section of the member, or on its "form" from which a "form factor" may be defined as a coefficient that evaluates the support given to the most stressed fiber by all the other fibers in a beam. Assuming a form factor of unity for the standard 2-in. square section tested with its sides vertical and horizontal (that is, one in which the elements in the layer adjacent to the most stressed are subjected to very nearly the same load as the most stressed) it may be

shown that the factor for a circular section of the same area is 1.15, while for the square section having its diagonals vertical and horizontal, it is 1.41.

It is obvious then that for an I section, the factor cannot be as great as 1.0, since the cut-out portion reduces the number of lightly stressed fibers that would otherwise offer support to the more highly stressed fibers in the flanges. By evaluating the supporting action which the fibers in the various parts of the cross-section exert, it should be possible to forecast the elastic limit stress and the stress at rupture for various types of sections. The empirical formula developed by the Forest Products Laboratory for evaluating this effect is:

$$F_e = 0.65 + 0.35\left[0.293\ (\alpha - \sin\alpha\cos\alpha)\frac{b - b'}{b} + \frac{b'}{b}\right] \qquad 12:1$$

which, with a slight change of constants, gives the modulus of rupture factor:

$$F_u = 0.50 + 0.50\left[0.293\ (\alpha - \sin\alpha\cos\alpha)\frac{b - b'}{b} + \frac{b'}{b}\right] \qquad 12:2$$

These equations are derived as follows: Assuming R_1 in Fig. 12 : 1 to be 1.0, the total area between curve A and the vertical axis, that is, the total supporting effect of the fibers is:

$$A = \tfrac{1}{2}\left[2.499 + \tfrac{3}{2} \times 2 - 0.928\ (\tfrac{3}{2})^2\right]$$

The area of the portion above the dashed line representing the flange to depth ratio is

$$A' = \tfrac{1}{2}\ (\alpha - \sin\alpha\cos\alpha)$$

The supporting ratio, K, shown on curve B is:

$$\frac{A'}{A} = 0.293\ (\alpha - \sin\alpha\cos\alpha)$$

These formulas represent the conditions when the depth of the compression flange is not more than 60 per cent of the total depth of the beam. Within this limit α is the angle the versed sine of which is t_c/h, measured in radians.

If the width of the flange of an I beam, or box beam, is b and the width of the web, b', the supporting ability of the compression chord is $\frac{A'}{A}\left(\frac{b - b'}{b}\right)$ times the supporting ability of a rectangular section of the breadth, b. The supporting ability of the web alone is $\frac{b'}{b}$ times

that of a rectangle of the breadth, b, so the total becomes $\frac{A'}{A}\left(\frac{b-b'}{b}\right) + \frac{b'}{b}$. From the strength properties of spruce it has already been noted that the total support given to the most stressed fibers increases the fiber stress at the elastic limit in flexure over that in pure compression by $\frac{6{,}200 - 4{,}000}{4{,}000}$ or 55 per cent. The increase for an I-beam section, or box section, is then, $0.55\left[\frac{A'}{A}\left(\frac{b-b'}{b}\right) + \frac{b'}{b}\right]$.

The ratio of the elastic limit stress in bending to the elastic limit of the material in direct compression will be 1.00 plus this quantity, and

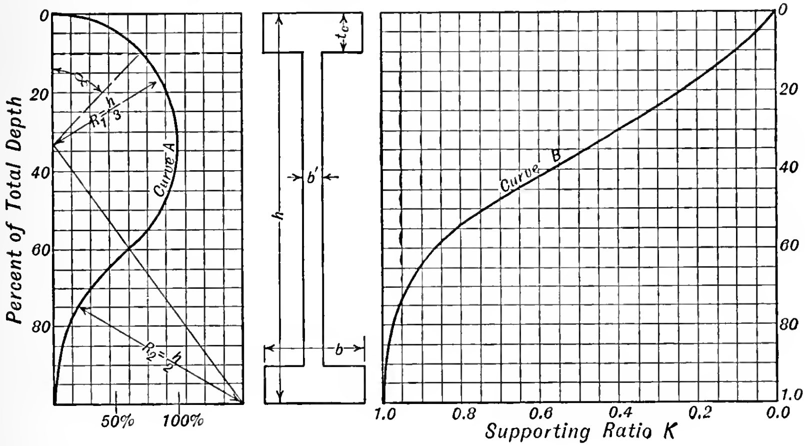


FIG. 12 : 1

the form factor will be this ratio divided by 1.55. Consequently, when the values of A and A' are substituted, the form factor for I-beam sections and box sections takes the shape of Formula 12 : 1.

Thus far, the assumption has been made that the stresses do not exceed the elastic limit of the material. When this limit is passed there is practically no theoretical justification for applying a formula similar to that given above to determine the effect of the form of a section on its properties. It has been found, however,[1] that if 0.50 and 0.50 be substituted for 0.65 and 0.35, a factor is obtained for the modulus of rupture that is in excellent accord with test results.

Neither of these formulas is directly applicable to beams the top and bottom surfaces of which are not at right angles to the vertical axis, a case that often occurs in wing-beams on account of their being beveled

[1] National Advisory Committee for Aeronautics, Report 181, page 4.

to conform to the wing contour. They do apply, however, to the properties of an equivalent section that is symmetrical about the vertical axis, the height of which is the mean height of the original section, and the width and flange areas of which equal those of the original section. By the use of these "equivalent" sections the form factor for practically any type of wooden beam likely to be used in an airplane wing may be obtained, and the modulus of rupture computed, thus determining the allowable stress for a beam subjected to pure flexure.

12 : 2. Stress in Members Under Combined Loadings — We have seen in Chapter IX that there are two expressions for computing the allowable stress in columns, one being Euler's formula, which is applicable as long as the fiber stress is less than about one-half the crushing strength, the other being Johnson's formula which applies when the fiber stress is greater.

In determining the allowable stresses in members under combined bending and compression it is necessary to use these formulas, each in its proper sphere, and, in addition, the form-factor formulas must be utilized to provide for the change in modulus of rupture with change in shape of the section.

Figure 12 : 4, page 249, contains curves based on these formulas and, by its use, the apparent allowable stress for any type of beam ordinarily encountered in airplane work may be readily computed for various combinations of axial and bending stresses.

The right half of the diagram contains curves that are practically self-explanatory. They are used to determine the modulus of rupture and elastic-limit-in-bending stresses for beams of various shapes. They are, in reality, the curves obtained by multiplying the form factors for beams of various shapes by the average value of the modulus of rupture of 9,400 lb. per sq. in., and the elastic-limit-in-bending value of 6,200 lb. per sq. in. The curves are plotted for various ratios of $\frac{t_c}{h}$ and $\frac{b'}{b}$ and give the desired modulus of rupture and elastic limit values in pounds per square inch instead of in terms of the form factor.

The curves in the left half of the diagram are concerned with the axial loads and are obtained in the following manner. The $\frac{L}{\rho}$ curves for the condition of zero bending, that is, $\frac{f_b}{f_t} = 0$, have values equal to those determined by Euler's or Johnson's formulas for the given values of $\frac{L}{\rho}$. Assuming that the axial load is kept constant and that the ratio of $\frac{f_b}{f_t}$ is increased to 0.2, which is the same as assuming that

the strut deflects under load, it is readily seen that the maximum stress, f_t, will be $\dfrac{1}{1 - 0.2} = 1.25$ times that developed under the critical load from Johnson's or Euler's formulas. Similarly, when $\dfrac{f_b}{f_t} = 0.5$, f_t is twice the stress under the critical load. The group of $\dfrac{L}{\rho}$ curves given in Fig. 12 : 4 were obtained by computing the critical stresses for each value of $\dfrac{L}{\rho}$ and multiplying each of these stresses by the proper factor to provide for the different ratios of $\dfrac{f_b}{f_t}$.

The other group, running approximately horizontal, indicates the intensity of stress at the elastic limit for various ratios of bending to total stress. These curves are plotted from values obtained as follows:

Assume a member having a rectangular section, or a form factor of unity. If it be subjected to a slight axial compression while a lateral load is being applied, the neutral surface will move toward the tension chord. The supporting action of the less stressed fibers will no longer be a maximum and the elastic limit stress will decrease. Considering the distance from the extreme compressive fiber to the neutral axis as the half height of a theoretical beam having but one chord and no webs, the ratio of chord depth to total depth of such a beam may be determined for various ratios of bending stress to total stress as follows. In Fig. 12 : 2, let

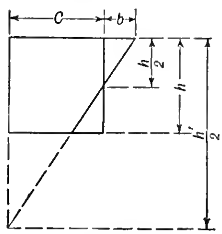


Fig. 12 : 2

$\dfrac{h'}{2}$ = the half height of the theoretical beam;

$\dfrac{h}{2}$ = the half height of the beam in question;

b = the total bending stress; and

c = the direct compressive stress.

Then, from similar triangles,

$$\frac{b}{\frac{h}{2}} = \frac{b + c}{\frac{h'}{2}}$$

or

$$\frac{h}{h'} = \frac{b}{b + c}.$$

In other words, the ratio of the flange depth to the total depth of the theoretical beam is the same as the ratio of the bending stress to the total stress. Since the theoretical beam has webs of zero thickness the form factor becomes

$$F_c = 0.65 + 0.35\ [0.293\ (\alpha - \sin\alpha\cos\alpha)]$$

The elastic limit stress for the combined load is then determined by taking the product of the elastic limit stress in ordinary bending, 6,200 lb. per sq. in., times the form factor of the theoretical beam for the particular ratio of bending stress to total stress. In the case of pure axial load, F_e becomes 0.65 which is the ratio between the elastic limit in compression and that in simple bending.

If the member being considered has a form factor, as, for instance, a box beam, the elastic limit stress in direct compression would remain unchanged, but that in bending would be lowered. To take a specific case, assume a form factor of 0.90. The elastic limit in bending would then be $0.90 \times 6{,}200 = 5{,}580$ lb. per sq. in., and the constants in the formula would be changed. The first would become $\dfrac{4{,}000}{5{,}580} = 0.717$, and the form-factor equation for stress at the elastic limit, which would be applied to the theoretical beam subjected to combined loads, would be,

$$F_e = 0.717 + 0.283 \times 0.293\ (\alpha - \sin\alpha\cos\alpha)$$

The elastic limit curves shown in Fig. 12 : 4 were drawn by the use of equations derived in this way.

It is now possible to consider the maximum load condition ror various ratios of direct to bending stresses. For the sake of simplicity consider a member in the Euler column class. Obviously, the Euler load is the maximum that can be obtained for a zero ratio of bending unit stress to total unit stress; and the stress must be less than the elastic limit stress in compression parallel to the grain since this is always less than one-half the ultimate for airplane spruce. Assume that the Euler stress for such a member is 2,000 lb. per sq. in. If the column deflected a little it would still carry the Euler load, but a bending stress would be introduced. Deflection would increase until the elastic limit was reached and the total stress, due to the axial compressive load and the bending stresses arising from the deflection, would follow the $\dfrac{L}{\rho} = 80$ curve in Fig. 12 : 4 until it intersected the elastic limit curve determined by the shape of the member. This intersection represents the stress in an axially loaded Euler column when deflected to the elastic limit. The stress at maximum load under eccentric or other combined

loading would always be somewhat greater than this elastic limit stress; hence this intersection serves as a starting point in the determination of the stress at maximum load. It has been found that the stress at maximum load will be intermediate between this point and the modulus of rupture and will lie on a straight line connecting these two points, at least within the limits of precision of the ordinary test.

The method of using the diagram can best be illustrated by determining the allowable load on an actual spar section, as, for example, that designed in Chapter XI to carry the loads in the rear upper spar of the wing structure analyzed in Chapter VII. Let us investigate the section of maximum moment in the outer bay where we have:

$$M_{1-2} = -4202 \text{ in.-lb.}$$
$$\frac{M_{1-2}}{w_1 j_1^2} = \frac{4202}{21134} = 0.199$$
$$j_1 = 48.46$$

From Fig. 12 : 3 we find the distance between the points of inflection in the bay to be 57.0 in., Fig. 12 : 3 being a nomogram for solving the equation for maximum moment on a pin-ended beam subjected to compression and flexure. The solution is obtained by drawing a straight line between the two known variables, $\frac{M_{1-2}}{w_1 j_1^2}$ and j_1, and noting its point of intersection with the axis representing the unknown, L_I. Having computed the radius of gyration for the beam, $\rho = \sqrt{\frac{I}{A}}$, we obtain $L_I/\rho = 42.2$

From the problem in Chapter XI we have a moment of 4202 in.-lb. producing a stress in bending in the extreme fiber of $f_b = \frac{4202 \times 2}{5.69} =$ 1480 lb. per sq. in. while that due to the axial load P is, $f_c = \frac{3150}{3.125} =$ 1010 lb. per sq. in. giving a total stress $f_t = 1480 + 1010 = 2490$ lb. per sq. in. Then $f_b/f_t = \frac{1480}{2490} = 0.595$. From Fig. 11 : 4 the thickness of the compression flange of the "equivalent" section is 0.75 in., the center line depth, the depth of the equivalent section, being 4.0 in. Then $\frac{t_c}{h} = \frac{0.75}{4.00} = 0.1875$. The web thickness is 0.5 in., and the overall width of the beam, 1.25 in., whence $\frac{b'}{b} = \frac{0.5}{1.25} = 0.4$. We now have all of the values necessary for use in Fig. 12 : 4 to determine the allowable stress. With the values of b'/b and t_c/h we enter

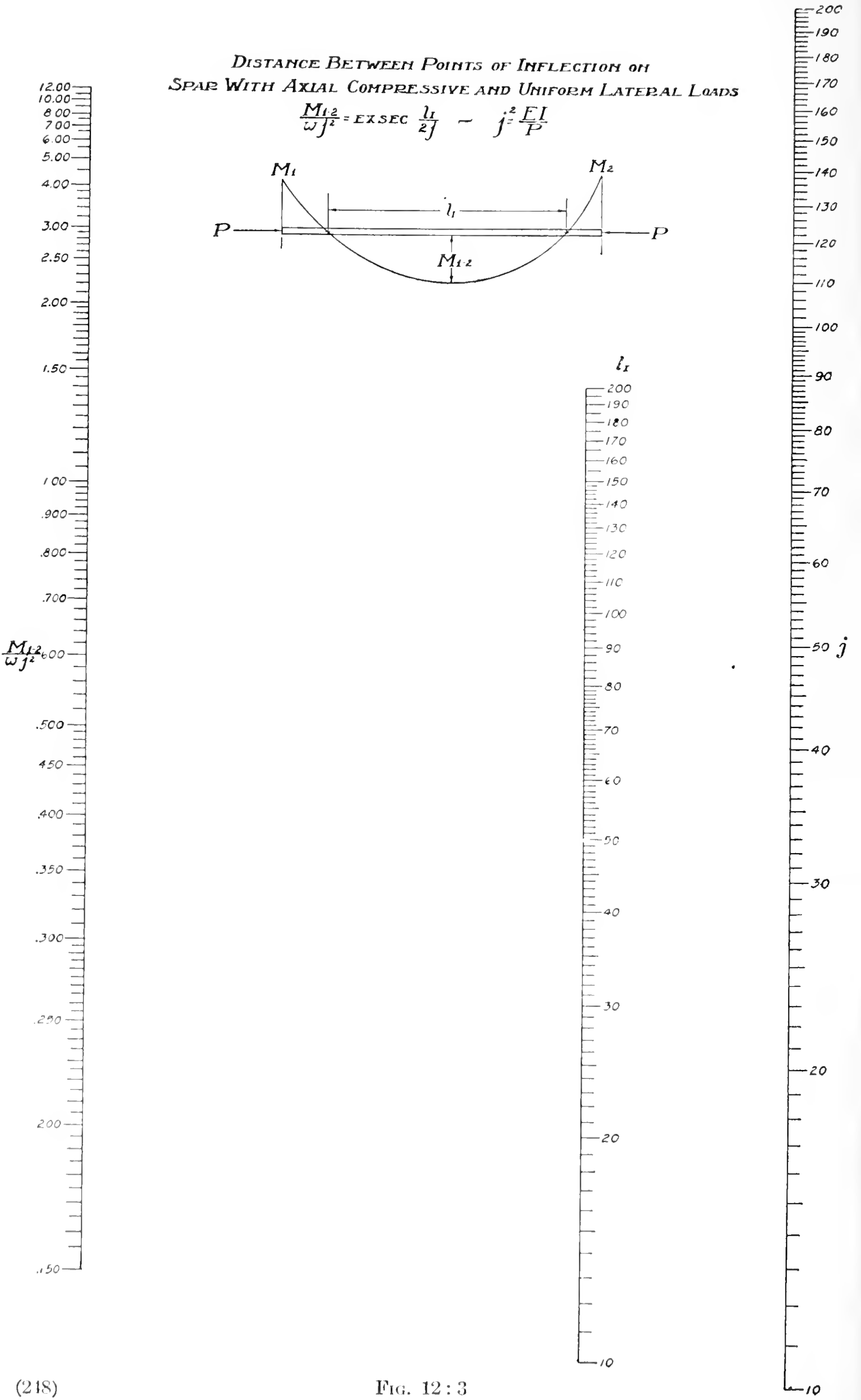
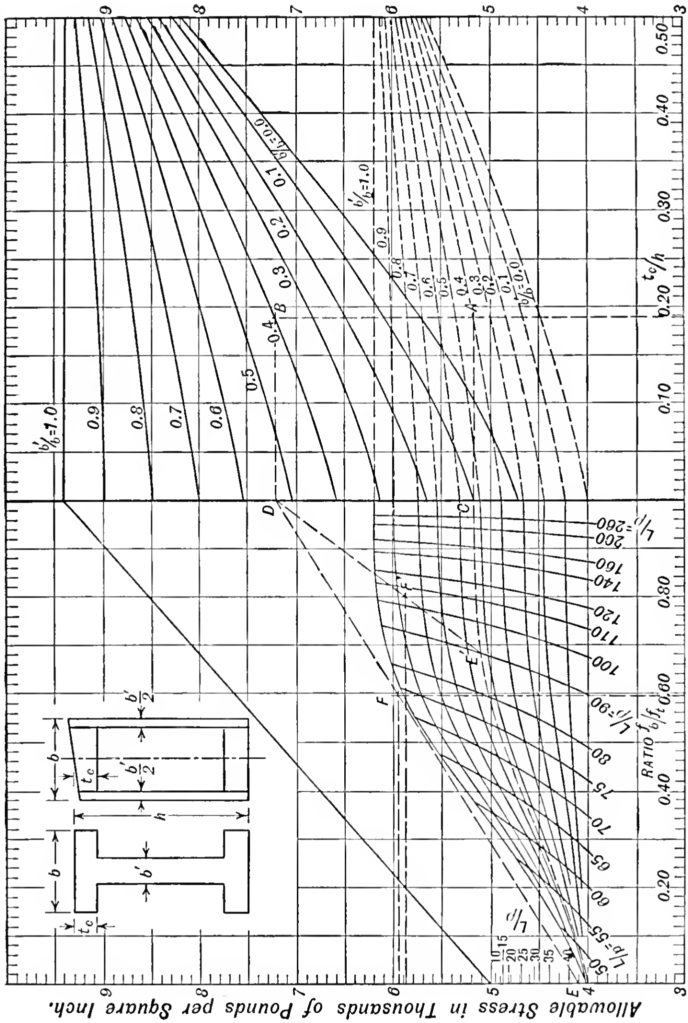

DISTANCE BETWEEN POINTS OF INFLECTION ON
SPAR WITH AXIAL COMPRESSIVE AND UNIFORM LATERAL LOADS
$\frac{M_{1\text{-}2}}{w j^2} = \text{EXSEC}\ \frac{l_1}{2j} \quad j^2 = \frac{EI}{P}$
M_1
M_2
l_1
P
P
$M_{1\text{-}2}$
$\frac{M_{1\text{-}2}}{w j^2}$
l_1
j

Fig. 12 : 3



Allowable Stress on Spruce Beams Subjected to Combined Flexure and Compression.

Fig. 12 : 4

the right half of the chart and locate points A and B at the intersections of the vertical drawn through $t_c/h = 0.1875$ with $b'/b = 0.4$. Point A, the intersection with the lower set of curves, represents the fiber stress at elastic limit in bending for this section. Point B gives the modulus of rupture for pure bending. Points A and B are projected horizontally to the central line in the chart, giving C and D. From C we draw a curve "parallel" to the elastic limit stress curves in the left half of the chart until it meets the $L/\rho = 42.2$ curve. It is to be noted that no such intersection may be obtained since the lowest L/ρ curve on the chart is for a value of 50. The value of L/ρ being below 50 indicates that the section of the spar between points of inflection is in the short column class and, if carrying its full column load, the intensity of stress would be beyond the elastic limit in compression. Hence we could have no deflection under such a load without the stress exceeding the elastic limit still further. Since no deflection is permissible and considering the method of constructing the L/ρ curves in the chart we would expect to find the point representing $L/\rho = 42.2$ on the $f_b/f_t = 0$ line as we do at E.

We now have point D representing the modulus of rupture in pure flexure, point E representing the allowable stress intensity in compression on the section of the spar considered as a column between the points of inflection. Since the actual load is a combination of bending and compression we would expect the spar to fail under a load intermediate between D and E. Assuming a rectilinear variation, — an assumption which has been substantiated by numerous tests — we draw a straight line between D and E and find its intersection with the abscissa representing $f_b/f_t = 0.595$, point F. The ordinate value for point F is, then, 5950 lb. per sq. in. This, being the allowable stress intensity on the most stressed fiber at the section investigated, is considerably more than the computed intensity of 2490 lb. per sq. in. and indicates that our trial section is sufficiently strong but is so greatly overstrength as to be unnecessarily heavy.

Had L/ρ been 90 and f_b/f_t been 0.80, point E would have been at E', point F at the intersection of DE' and $f_b/f_t = 0.80$, F', and the allowable unit stress would have been reduced to 5890 lb. per sq. in.

In using this chart the slenderness ratio should be that between points of inflection. When the ends of the member are restrained, this need not be computed with excessive precision and the following practice is now well established.

In any span of a continuous or restrained beam, L may be taken as the distance between points of inflection under side load alone or may be obtained for the combined loading by using Fig. 12 : 3. The latter

is preferred. At points of support L is taken as the distance between adjacent points of inflection except at the end supports where it is assumed to be twice the distance from the given support to the outer point of inflection. In computing ρ, filler blocks may be neglected and for tapered spars the average value may be used. Filler blocks, if rigidly attached to the spar, may be included in the section when computing its properties and determining f_b, f_c and f_t.

When tension and bending occur in combination the allowable stress is taken as the modulus of rupture for the section in pure flexure.

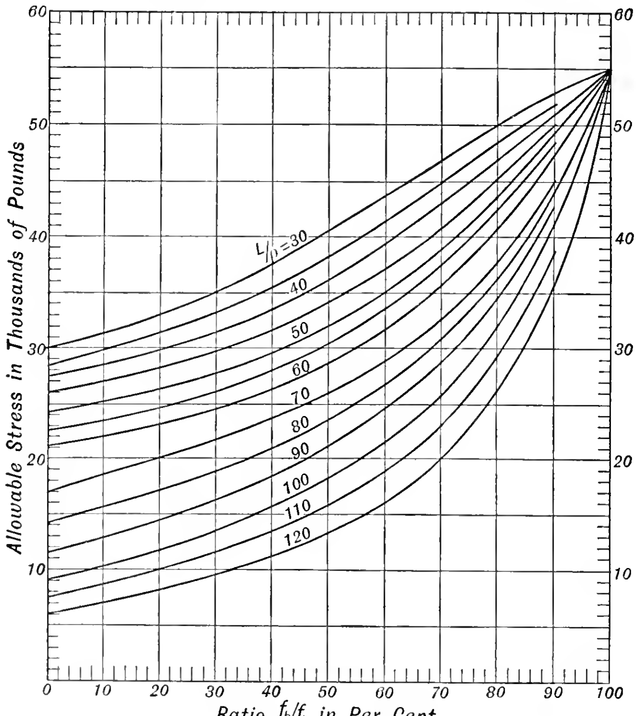
In some cases in which axial compression is combined with bending there are no points of inflection between which the slenderness ratio can be logically measured. One such case is that of a spar in an internally braced cantilever monoplane that is subjected to large bending moments from the beam loads on the wing and to axial compression due to its being a member of the internal drag truss. A reasonable method of handling this case, and others in which the unit stress due to axial load is small in comparison to the unit stress due to bending is to assume the allowable total stress, f_t, equal to the modulus of rupture as obtained from the right hand side of Fig. 12 : 4 multiplied by the ratio of f_b/f_t.

In a few cases spans will be found where the stress due to axial load is a large part of the total stress but in which there are no points of inflection on account of the large size of the end moments as compared to the length of the span and the magnitude of the lateral load. This condition brings up two questions, "Where should the point of inflection be assumed when investigating the strength of the end sections?" and "What distance between points of inflection should be assumed when investigating sections near the center of the span?" A reasonable answer to the first is to assume a point of inflection where the curve of bending moment in the span changes slope, i.e., at the point of zero shear in the span. A reasonable answer to the second is to assume the distance in question to be the length of the span.

12 : 3. Metal Members Subjected to Combined Loads — While data on wood sections are reasonably complete and satisfactory, but few exist on the strength of metal members subjected to combined loadings. For tubular members the assumption is commonly made that the maximum allowable stress on a member subjected to combined bending and axial loads equals the ultimate tensile strength of the material. Such members may therefore be designed to carry a combination of tension and bending without recourse to complicated formulas or expressions.

For combined bending and compression the situation is more difficult

since the ratio of stress due to bending to total stress is involved. As shown by Fig. 12 : 5, which is based on test data, the allowable stress in pure bending for aluminum alloy tubes is 55,000 lb. per sq. in. This corresponds to the ultimate tensile strength for this material. The allowable stress per square inch in plain compression varies with the slenderness ratio and agrees satisfactorily with the values given by the



Allowable Stress on Aluminum Alloy Tubes Subjected to Combined Flexure and Compression.[1]

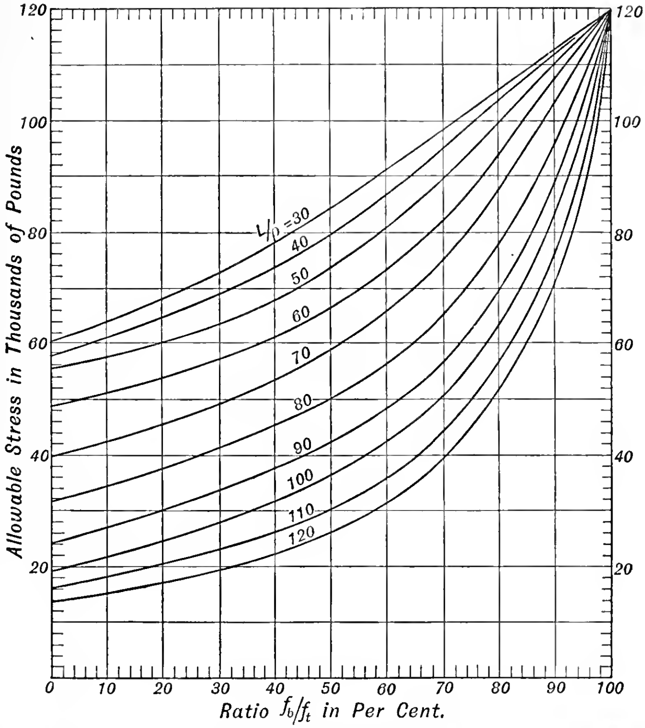
Fig. 12 : 5

straight line-Euler column curve when it is noted that the values given in Fig. 12 : 5 are minima rather than averages. For combinations of stress intermediate between pure compression and pure flexure the allowable total stress as found from $f_t = \frac{P}{A} + \frac{My}{I}$, where M includes the effect of secondary bending moments, varies with L/ρ and with the ratio of bending stress to total stress. In using this chart the

[1] Replotted from data obtained by U. S. Bureau of Standards for Bureau of Aeronautics, U. S. Navy.

symbols have the same significance as in Fig. 12 : 4, L being the distance between pin joints or between points of inflection on restrained members. Having computed the values of f_b/f_t and L/ρ the determination of the allowable stress becomes a matter of finding the ordinate to the L/ρ curve at the appropriate value of f_b/f_t.

It is probable that the values for mild carbon steel (S. A. E. 1025) tubes would be very nearly the same as those given in Fig. 12 : 5 for



Allowable Stress on Chrome-molybdenum Steel Tubes Subjected to Combined Flexure and Compression.[1]

FIG. 12 : 6

aluminum alloy since the strength properties of these two materials are approximately the same. It is, therefore, recommended that Fig. 12 : 5 be used for mild steel as well as aluminum alloy, at least until such time as test data are obtained to substantiate other values.

Figure 12 : 6 gives allowable stress values for chrome molybdenum

[1] Replotted from data obtained by U. S. Bureau of Standards for Bureau of Aeronautics, U. S. Navy.

steel tubing. It is interesting to note that the modulus of rupture in pure bending is materially above the rated tensile strength of chrome molybdenum steel being 120,000 instead of 95,000 lb. per sq. in.

For heat-treated steels recourse must be had to a formula which, though justified by use in many successful airplanes, has never been completely vindicated by test data. The formula assumes a rectilinear variation in allowable stress for values of f_b/f_t between 0.0 and 1.0. It is

$$F_a = \frac{f_b}{f_t}(R - C) + C, \text{ where,}$$

F_a represents the allowable intensity of stress

R is the modulus of rupture or ultimate tensile strength of the material

C is the allowable unit compressive strength on the member considered as a column of length equal to the distance between pin joints or points of inflection.

It is to be noted that curves corresponding to this expression would, if drawn on Figs. 12 : 5 or 12 : 6, lie considerably above the curves actually obtained by test except for the tubes having low values of L/ρ. The assumption that C may be taken as the yield point stress for the material would raise such curves still further and increase the discrepancy between the values given by the formula and those obtained by test. This assumption is, in the light of the data on aluminum alloy and chrome molybdenum tubes, untenable and the use of the yield point stress for C is not recommended.

Metal sections other than tubes generally fail by buckling of the compression chord and do not fail by rupture of the fibers as is the case with a wooden member. Such buckling often occurs in a plane at right angles to the plane of flexure of the beam, hence must be guarded against by providing external lateral bracing or by proportioning the section so that its moment of inertia about an axis in the plane of bending is large. Occasionally thin gauge materials fail by buckling or crinkling of the metal and failure in such cases is generally localized at or near the most highly stressed section. Although analytical processes are available for the determination of the critical stress intensities at which buckling will occur they involve a number of formidable mathematical equations and the results obtained from their solution are none too satisfactory even for the less complicated structural shapes. The methods do indicate, however, that the tendency toward lateral buckling may be reduced if the section has a large moment of inertia about an axis in the plane of bending and if it is inherently stiff tor-

sionally. The validity of these indications has been confirmed by tests on metal spar sections which show that channel trusses the web members of which are of the same width as the chords have better lateral strength than similar trusses in which the web system is composed of small channels or angles riveted to the legs of the chord members but not extending across the entire width of the section. Similarly, box sections having internal diaphragms which are rigidly fastened to the webs have better strength characteristics as regards lateral failures than plate girder types having a single web plate which is torsionally weak.

Because of the tendency for metal members to buckle, a tendency which is greatly affected by the shape of the section, it is well to build up and test a full scale member of the size and type proposed before incorporating it in an airplane. The following assumptions will be found to be of assistance in proportioning tentative designs, the assumptions to be substantiated and the final design determined by test.

For the chord members of trussed spars assume points of inflection at each panel point of the truss for bending in the plane of the truss. For lateral buckling investigate the compression chord as a column subjected to the average axial load which it carries between points of inflection assumed at each drag truss panel point. If the ribs have adequate strength to reduce the tendency toward lateral buckling, the distance between points of inflection may be reduced to one-half the distance between drag struts or compression ribs. Where corrugated covering is used the tendency toward lateral buckling may be neglected entirely. The allowable loads for seamless tubular chords when investigated for the above conditions may be found from Figs. 9 : 10 to 9 : 19. For aluminum alloy channels use Figs. 9 : 21 and 9 : 22. For other shapes or materials the strengths should be obtained by tests.

Web members should be designed under the assumption that they are pin jointed at each end. If the joint is such as to produce bending in the web members these members must be investigated for the combined stress to which they will be subjected as small eccentricities at the ends of such members will greatly reduce their strength.

The chord members of aluminum alloy spars of the girder or box type should be designed for a maximum stress under combined bending and compression of not more than from 30,000 to 35,000 lb. per sq. in. For steel beams the allowable stress will vary with the alloy or heat treatment and shape used, hence definite figures cannot be offered.

The strength of continuous webs and the location and size of web stiffeners may be determined by the standard methods for girder design

as outlined in Art. 9 : 10, proper account being taken of differences in moduli of elasticity. For spars having lightening holes in the webs the shear strength must be determined by test.

PROBLEMS

12 : 1. Determine the allowable stress in pure bending on a spruce spar having the cross-sectional dimensions shown in Fig. 10 : 7. Assuming that the webs are braced by spruce stiffeners at 5-in. intervals, what is the allowable longitudinal shearing stress on the webs? It is desired to test such a spar so that it will fail simultaneously in shear and bending. How long a span would be required if a single load, W, is applied at mid-span? What is the magnitude of load W?

12 : 2. Determine the allowable stresses and margins of safety for the spruce spar section investigated in **Prob. 11 : 1.**

12 : 3. Determine the allowable stress and margin of safety for the chrome molybdenum steel tube investigated in **Prob. 11 : 2.**

CHAPTER XIII

DEFLECTIONS

Deflections of beams and trusses must be computed for two purposes, to determine the stiffness of the structure, and to enable a computation to be made of the secondary bending moments induced by the simultaneous action of axial compression and bending. The method of computation to be used in any given case depends on the character of the structure, the type of loading, and the number and location of the points the deflections of which are desired. In this chapter a number of methods of computation are described so the student who has mastered them will be able to handle nearly any deflection problem he may meet in the easiest manner.

13 : 1. Deflection Formulas Derived by the Method of Successive Integration — All texts on Mechanics of Materials show how the deflection of a beam of constant cross-section subjected to loads which produce only pure bending may be determined by two successive integrations of the equations for curves of M/EI between the limits to which those equations apply. As the student is assumed to be familiar with this method for obtaining deflections no attempt is made in this volume to prove its validity. However, for reference purposes, formulas for deflections and for some of the more useful properties of the elastic curve are given in Table 13 : 1 for the more common conditions of loading occurring in airplane design. These formulas give the deflection and slope at any section of a beam subjected to transverse loads only but they are not applicable to beams carrying axial loads, except for the computation of "primary" deflections. When the loading considered is of a complex character but can be broken up into two or more of the loadings for which formulas are given, the formula for the combination may be obtained by combining those for the constituent loads.

The nomenclature and conventions for signs used in these formulas are the same as those listed in Art. 11 : 5 and used generally throughout this volume. Only a few of the terms used need be discussed here. In the formulas of this article the locations of the various quantities are expressed by subscripts. When a quantity changes abruptly at a section, its value an infinitesimal distance to the left of the section is indicated by a minus sign in the subscript, and that at an infinitesimal distance to the right by a plus sign. Thus S_{+1} represents the shear just to the right of section 1.

In addition to giving the equations for the curves of load, shear, moment, slope, and deflection, and the values of these quantities at the more important sections, in a number of cases the area under the moment curve, A_m, and the location of the centroid of that area are included in the table.

The deflection formulas of this article apply strictly only to the deflection due to bending of integral homogeneous beams such as extruded aluminum alloy I beams. As the standard values of E for wood, steel, and aluminum alloy used in design are intentionally somewhat less than the true ratio of stress to strain below the elastic limit in order to make some allowance for shear deflections, these formulas may be used in practice for wood box and I beams as well as integral extruded or rolled metal sections with continuous webs without lightening holes.

If a beam is built up of metal sections connected by rivets, the deflections will be increased due to play in the joints and the concentrations of internal stress caused by the lack of continuous connection between the parts of the beam. In such cases it is recommended that the product of the geometrical moment of inertia of the section and the modulus of elasticity of the material be multiplied by 0.85, when using these formulas.

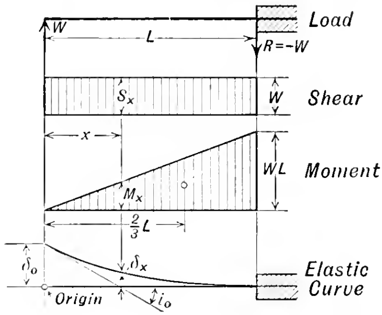
If a beam has its web pierced by lightening holes, the shear deflections will be very greatly increased. If possible, such beams should be considered as trusses when computing their deflections. If the shape of the lightening holes is such as to prevent this, it is best to determine the effective value of EI by test. For preliminary work, however, an approximate value for the effective EI can be obtained by multiplying the product of I of the section and E of the material by 0.75. This last case is most commonly represented by beams with round or oblong lightening holes.

The formulas of Table 13 : 1 cannot be used for computing the deflections of trusses unless the effective EI values are obtained by tests. They are also inapplicable to beams of non-uniform cross-section. Other methods, however, have been developed for computing deflections which can be used for these cases and which are often the easiest to use with beams of uniform cross-section.

TABLE 13 : 1

SHEAR, MOMENT AND DEFLECTION FORMULAS FOR BEAMS

CASE 1. CANTILEVER WITH CONCENTRATED LOAD AT FREE END.



$R = -W$ $\qquad S_x = +W$

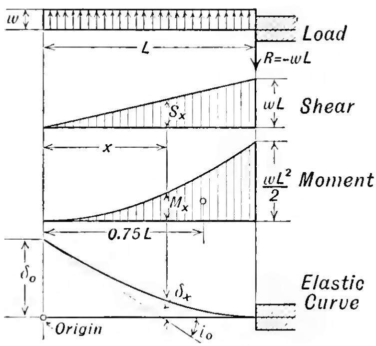
$M_x = +Wx$ $\qquad i_x = \dfrac{-W}{2\,EI}(L^2 - x^2)$

Elastic curve $\delta = \dfrac{W}{6\,EI}(2\,L^3 - 3\,L^2x + x^3)$

$i_0 = -\dfrac{WL^2}{2\,EI}$ $\qquad \delta_0 = +\dfrac{WL^3}{3\,EI}$

$A_m = \dfrac{WL^2}{2}$

CASE 2. CANTILEVER WITH UNIFORM LOAD.



$R = -wL$ $\qquad S_x = +wx$

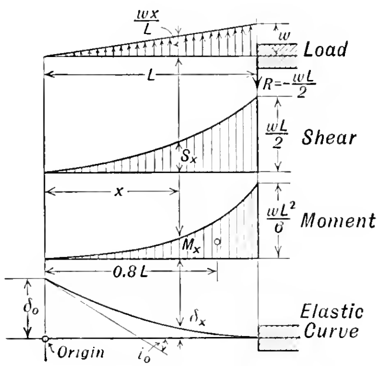
$M_x = +\dfrac{wx^2}{2}$ $\qquad i_x = -\dfrac{w}{6\,EI}(L^3 - x^3)$

Elastic curve $\delta = +\dfrac{w}{24\,EI}(3\,L^4 - 4\,L^3x + x^4)$

$i_0 = -\dfrac{wL^3}{6\,EI}$ $\qquad \delta_0 = +\dfrac{wL^4}{8\,EI}$

$A_m = \dfrac{wL^3}{6}$

CASE 3. CANTILEVER WITH UNIFORMLY VARYING LOAD.



$$R = -\frac{wL}{2} \qquad S_x = +\frac{wx^2}{2\,L}$$

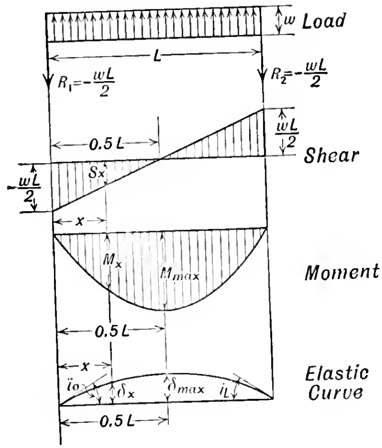
$$M_x = +\frac{wx^3}{6\,L} \qquad i_x = -\frac{w}{24\,EI}(L^4 - x^4)$$

Elastic curve $\delta = +\frac{w}{120\,EI}(4\,L^5 - 5\,L^4x + x^5)$

$$i_0 = -\frac{wL^3}{24\,EI} \qquad \delta_0 = +\frac{wL^4}{30\,EI}$$

$$A_m = \frac{wL^3}{24}$$

CASE 4. SIMPLE BEAM WITH UNIFORM LOAD.



$$R_1 = -\frac{wL}{2} \qquad R_2 = -\frac{wL}{2}$$

$$S_x = -\frac{wL}{2} + wx \qquad M_x = -\frac{wLx}{2} + \frac{w}{2}x^2$$

$$i_x = +\frac{w}{24\,EI}(L^3 - 6\,Lx^2 + 4\,x^3) \qquad M_{\max} = -\frac{wL^2}{8}$$

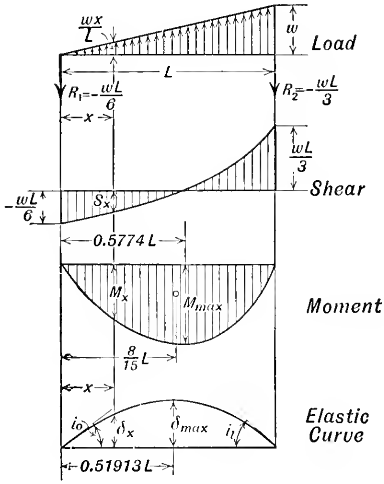
$$i_0 = +\frac{wL^3}{24\,EI} \qquad i_L = -\frac{wL^3}{24\,EI}$$

Elastic curve $\delta = \frac{wx}{24\,EI}(L^3 - 2\,Lx^2 + x^3)$

or $$\delta = \frac{x(x-L)}{EI}\left[\frac{S_{+1}}{6}(x+L) + \frac{w}{24}(x^2 + xL + L^2)\right]$$

$$\delta_{\max} = \frac{5\,wL^4}{384\,EI} \text{ at } x = \frac{L}{2} \qquad A_m = -\frac{wL^3}{12}$$

Case 5. Simple Beam with Uniformly Varying Load Increasing to the Right.



$$R_1 = -\frac{wL}{6} \qquad\qquad R_2 = -\frac{wL}{3}$$

$$S_x = -\frac{w}{2\,L}\left(\frac{L^2}{3} - x^2\right) \qquad\qquad M_x = -\frac{wx}{6\,L}(L^2 - x^2)$$

$$i_x = \frac{w}{360\,EIL}(7\,L^4 - 30\,L^2x^2 + 15\,x^4) \qquad M_{\max} = -0.064\,wL^2$$

$$i_0 = +\frac{7\,wL^3}{360\,EI} \qquad\qquad i_L = -\frac{wL^3}{45\,EI}$$

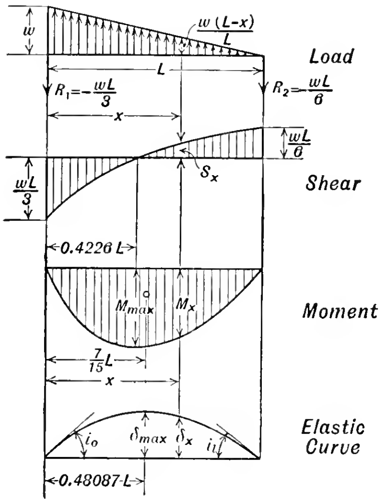
$$\text{Elastic curve } \delta = \frac{wx}{360\,EIL}(7\,L^4 - 10\,L^2x^2 + 3\,x^4)$$

or

$$\delta = \frac{x(x - L)}{EI}\left[\frac{S_{+1}}{6}(x + L) + \frac{w}{120\,L}(x + L)(x^2 + L^2)\right]$$

$$\delta_{\max} = 0.00632\,\frac{wL^4}{EI} \text{ at } x = 0.51913\,L \qquad\qquad A_m = -\frac{wL^3}{24}$$

Case 6. Simple Beam with Uniformly Varying Load Increasing to the Left.



$$R_1 = -\frac{wL}{3} \qquad R_2 = -\frac{wL}{6}$$

$$S_x = -\frac{w}{L}\left(\frac{L^2}{3} - Lx + \frac{x^2}{2}\right) \qquad M_x = -\frac{wx}{6\,L}(2\,L^2 - 3\,Lx + x^2)$$

$$i_x = \frac{w}{360\,EIL}(8L^4 - 60L^2x^2 + 60Lx^3 - 15x^4) \qquad M_{\max} = 0.064\,wL^2$$

$$i_0 = +\frac{wL^3}{45\,EI} \qquad i_L = -\frac{7\,wL^3}{360\,EI}$$

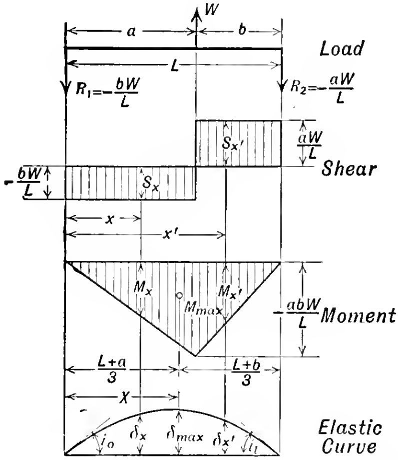
$$\text{Elastic curve } \delta = \frac{wx}{360\,EIL}(8\,L^4 - 20\,L^2x^2 + 15\,Lx^3 - 3\,x^4)$$

or

$$\delta = \frac{x(x - L)}{EI}\left[\frac{S_{+1}}{6}(x + L) + \frac{w}{30\,L}(x + L)(x^2 + L^2) - \frac{wx^3}{24\,L}\right]$$

$$\delta_{\max} = 0.00632\,\frac{wL^4}{EI} \text{ at } x = 0.48087\,L \qquad A_m = -\frac{wL^3}{24}$$

CASE 7. SIMPLE BEAM WITH SINGLE CONCENTRATED LOAD.



$$R_1 = -\frac{bW}{L} \qquad R_2 = -\frac{aW}{L}$$

x refers to distances from left end less than a.
x' refers to distances from left end greater than a.

$$S_x = -\frac{bW}{L} \qquad S_{x'} = +\frac{aW}{L}$$

$$M_x = -\frac{Wbx}{L} \qquad M_{x'} = -\frac{Wa}{L}(L - x') \qquad M_{\max} = -\frac{Wab}{L}$$

$$i_x = \frac{Wb}{6\,EIL}(L^2 - b^2 - 3\,x^2) \qquad i_{x'} = \frac{Wa}{6\,EIL}(2\,L^2 - 6\,Lx + 3\,x^2 + a^2)$$

Elastic curve $x < a$ $\quad \delta = \frac{Wbx}{6\,EIL}(L^2 - b^2 - x^2)$

or $\quad \delta = \frac{x(x-L)}{EI}\left[\frac{S_{+1}}{6}(x + L) - \frac{Wb^3}{6\,L(x - L)}\right]$

Elastic curve $x' > a$ $\quad \delta' = \frac{Wa(L - x')}{6\,EIL}(2\,Lx - a^2 - x'^2)$

or $\quad \delta' = \frac{x'(x'-L)}{EI}\left[\frac{S_{+1}}{6}(x' + L) - \frac{W}{6(x' - L)}\left(\frac{b^3}{L} - \frac{(x'-a)^3}{x'L}\right)\right]$

$$i_0 = \frac{Wb}{6\,EIL}(L^2 - b^2) \qquad i_L = \frac{Wa}{6\,EIL}(a^2 - L^2)$$

$$i_a = \frac{Wab}{3\,EIL}(L - 2a) = \frac{Wab}{3\,EIL}(2\,b - L)$$

If $a > b$, $\delta_{\max} = \frac{WbX^3}{3\,EIL}$ where $X = \sqrt{\frac{L^2 - b^2}{3}}$

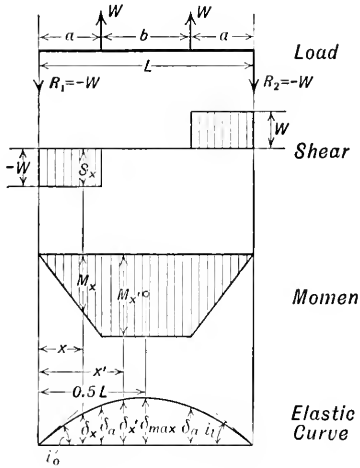
If $a < b$, $\delta_{\max} = \frac{Wa(L - X)^3}{3\,EIL}$ where $X = L - \sqrt{\frac{L^2 - a^2}{3}}$

$$A_m = -\frac{Wab}{2}$$

From R_1 to c.g. of M curve is $x_0 = \frac{L + a}{3}$

From R_2 to c.g. of M curve is $L - x_0 = \frac{L + b}{3}$

CASE 8. SIMPLE BEAM WITH TWO EQUAL, SYMMETRICALLY PLACED CONCENTRATED LOADS.



$R_1 = -W \qquad R_2 = -W \qquad S_x = -W \qquad S_{x'} = 0$

$M_x = -Wx \qquad M_{x'} = -Wa \qquad M_{\max} = -Wa$

$$i_x = \frac{W}{2\,EI}(La - a^2 - x^2) \qquad i_{x'} = \frac{Wa}{2\,EI}(L - 2\,x')$$

$$i_0 = \frac{Wa}{2\,EI}(L - a) \qquad i_a = \frac{Wab}{2\,EI} \qquad i_L = -\frac{Wa}{2\,EI}(L - a)$$

Elastic curve $x < a$ $\quad \delta = \dfrac{Wx}{6\,EI}(3\,La - 3\,a^2 - x^2)$

or
$$\delta = \frac{x(x - L)}{EI}\left[\frac{S_{+1}}{6}(x + L) - \frac{W}{6\,(x - L)}(a^2 + ab + b^2)\right]$$

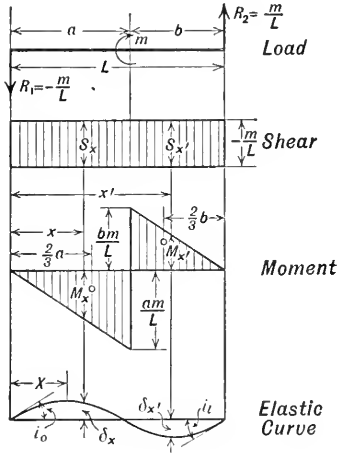
Elastic curve $a < x < (L - a)$ $\quad \delta' = \dfrac{Wa}{6\,EI}(3\,Lx' - 3\,x'^2 - a^2)$

or
$$\delta' = \frac{x'(x' - L)}{EI}\left[\frac{S_{+1}}{6}(x' + L) - \frac{W}{6\,(x' - L)}\left(a^2 + ab + b^2 - \frac{(x' - a)^3}{x'}\right)\right]$$

$$\delta_a = \frac{Wa^2}{6\,EI}(3\,L - 4\,a) \qquad A_m = -Wa(L - a)$$

$$\delta_{\max} = \frac{Wa}{24\,EI}(3\,L^2 - 4\,a^2) \text{ when } x' = \frac{L}{2}$$

Case 9. Simple Beam with External Moment Applied in the Span.



$$R_1 = -\frac{m}{L} \qquad R_2 = +\frac{m}{L} \qquad S_x = -\frac{m}{L} \qquad S_{x'} = -\frac{m}{L}$$

$$M_x = -\frac{mx}{L} \qquad M_{x'} = m - \frac{mx'}{L} \qquad M_{\max} = -\frac{am}{L} \text{ or } +\frac{bm}{L}$$

$$i_x = \frac{m}{6\,EIL}(L^2 - 3\,b^2 - 3\,x^2) \qquad i_{x'} = \frac{m}{6\,EIL}(6\,Lx' - 2\,L^2 - 3\,a^2 - 3\,x'^2)$$

$$i_0 = \frac{m}{6\,EIL}(L^2 - 3\,b^2) \qquad i_a = \frac{m}{3\,EIL}(3\,ab - L^2) \qquad i_L = \frac{m}{6\,EIL}(L^2 - 3\,a^2)$$

Elastic curve $x < a$ $\quad \delta = \dfrac{mx}{6\,EIL}(L^2 - 3\,b^2 - x^2)$

or $\quad \delta = \dfrac{x(x - L)}{EI}\left[\dfrac{S_{+1}}{6}(x + L) - \dfrac{mb^2}{2\,L(x - L)}\right]$

Elastic curve $x' > a$ $\quad \delta' = \dfrac{m(L - x')}{6\,EIL}(x'^2 + 3\,a^2 - 2\,Lx')$

or $\quad \delta' = \dfrac{x'(x' - L)}{EI}\left[\dfrac{S_{+1}}{6}(x' + L) - \dfrac{m}{2(x' - L)}\left(\dfrac{b^2}{L} - \dfrac{(x' - a)^2}{x'}\right)\right]$

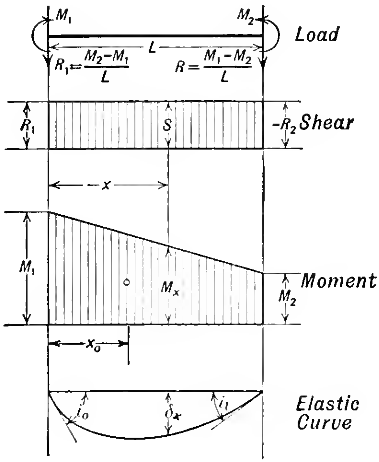
$$A_m = \frac{m}{2}(b - a)$$

Distance from R_1 to c.g. of M curve $= \dfrac{L^2 - 3\,a^2}{3\,(L - 2\,a)} = \dfrac{2\,a}{3} + \dfrac{b^2}{3\,(b - a)}$.

If $a > b$ $\quad \delta_{\max} = \dfrac{mX^3}{3\,EIL}$ where $X = \sqrt{\dfrac{L^2}{3} - b^2}$

If $a < b$ $\quad \delta_{\max} = \dfrac{m(X - L)^3}{3\,EIL}$ where $X = L - \sqrt{\dfrac{L^2}{3} - a^2}$

CASE 10. RESTRAINED BEAM WITHOUT SIDE LOADS.



$$R_1 = \frac{M_2 - M_1}{L} \qquad R_2 = \frac{M_1 - M_2}{L}$$

$$S = \frac{M_2 - M_1}{L} \qquad M_x = M_1 + \frac{M_2 - M_1}{L}x$$

$$i_x = \frac{M_2 - M_1}{6\,EIL}(3\,x^2 - L^2) + \frac{M_1}{2}(2\,x - L)$$

or $$i_x = \frac{S_{+1}}{6\,EI}(3\,x^2 - L^2) + \frac{M_1}{2\,EI}(2\,x - L)$$

$$i_0 = -\frac{L}{6\,EI}(2\,M_1 + M_2) \qquad i_L = \frac{L}{6\,EI}(M_1 + 2\,M_2)$$

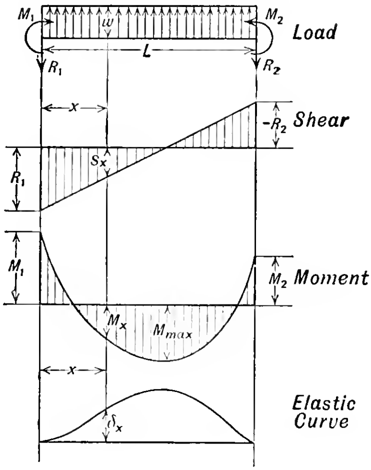
$$\delta = \frac{x(x - L)}{EI}\left[\frac{M_1}{2} + \frac{M_2 - M_1}{6\,L}(x + L)\right]$$

or $$\delta = \frac{x(x - L)}{EI}\left[\frac{M_1}{2} + \frac{S_{+1}}{6}(x + L)\right]$$

$$A_m = \frac{(M_1 + M_2)}{2}L$$

From R_1 to c.g. of moment curve is $x_0 = \dfrac{M_1 + 2\,M_2}{M_1 + M_2}\cdot\dfrac{L}{3} = \dfrac{L}{3} + \dfrac{M_2L}{3\,(M_1 + M_2)}$

Case 11. Restrained Beam with Uniform Load.



$$R_1 = \frac{M_2 - M_1}{L_1} - \frac{wL}{2} \qquad R_2 = \frac{M_1 - M_2}{L} - \frac{wL}{2}$$

$$S_x = S_1 + wx = \frac{M_2 - M_1}{L} - \frac{wL}{2} + wx$$

$$M_x = M_1 + S_{+1}x + \frac{wx^2}{2}$$

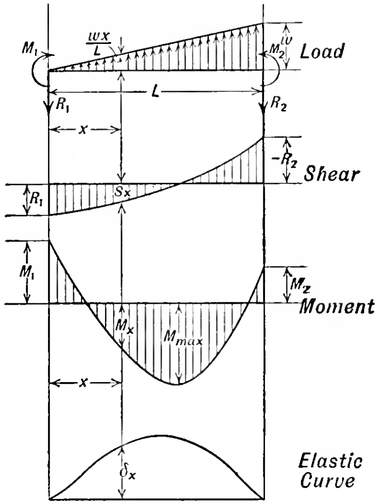
Elastic curve $\delta = \frac{x(x-L)}{EI}\left[\frac{M_1}{2} + \frac{S_{+1}}{6}(x+L) + \frac{w}{24}(x^2 + xL + L^2)\right]$

$$EI\,\delta_{L/2} = -\frac{L^2}{4}\left(\frac{M_1}{2} + \frac{S_{+1}L}{4} + \frac{7\,L^2w}{96}\right)$$

If $M_1 = M_2$

$$\delta_{\max} = \delta_{L/2} = -\frac{L^2}{4}\left(\frac{M_1}{2} - \frac{5\,wL^2}{96}\right)$$

Case 12. Restrained Beam with Uniformly Varying Load Increasing to the Right.

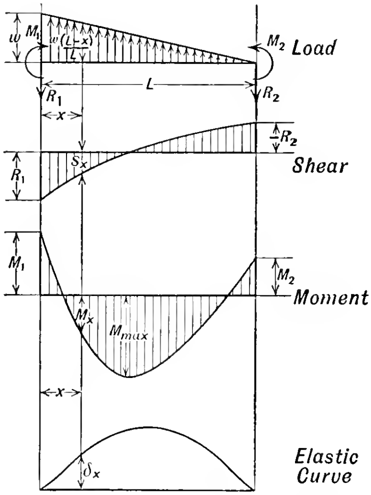


$$R_1 = \frac{M_2 - M_1}{L} - \frac{wL}{6} \qquad R_2 = \frac{M_1 - M_2}{L} - \frac{wL}{3}$$

$$S_x = S_{+1} + \frac{wx^2}{2L} \qquad M_x = M_1 + S_{+1}x + \frac{wx^3}{6L}$$

$$\text{Elastic curve } \delta = \frac{x(x-L)}{EI}\left[\frac{M_1}{2} + \frac{S_{+1}}{6}(x+L) + \frac{w}{120L}(x+L)(x^2+L^2)\right]$$

CASE 13. RESTRAINED BEAM WITH UNIFORMLY VARYING LOAD INCREASING TO THE LEFT.



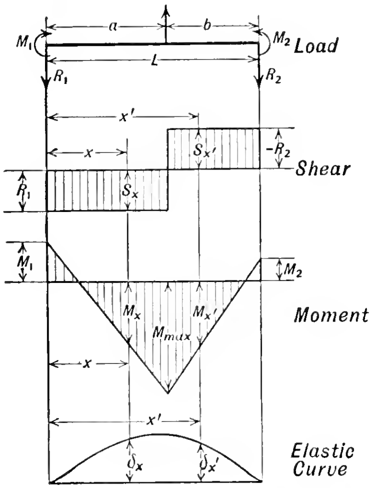
$$R_1 = \frac{M_2 - M_1}{L} - \frac{wL}{3} \qquad R_2 = \frac{M_1 - M_2}{L} - \frac{wL}{6}$$

$$S_x = S_{+1} + wx - \frac{wx^2}{2L} \qquad M_x = M_1 + S_{+1}x + \frac{wx^2}{2} - \frac{wx^3}{6L}$$

$$\text{Elastic curve } \delta = \frac{x(x-L)}{EI}\left[\frac{M_1}{2} + \frac{S_{+1}}{6}(x+L) + \frac{w}{30L}(x+L)(x^2+L^2) - \frac{wx^3}{24L}\right]$$

$$\text{or} \quad \delta = \frac{x(x-L)}{EI}\left[\frac{M_1}{2} + \frac{S_{+1}}{6}(x+L) + \frac{w}{24}(x^2+xL+L^2) - \frac{w}{120L}(x+L)(x^2+L^2)\right]$$

CASE 14. RESTRAINED BEAM WITH SINGLE CONCENTRATED LOAD.



$$R_1 = \frac{M_2 - M_1}{L} - \frac{Wb}{L} \qquad R_2 = \frac{M_1 - M_2}{L} - \frac{Wa}{L}$$

$$S_x = S_{+1} \qquad S_{x'} = S_{+1} + W$$

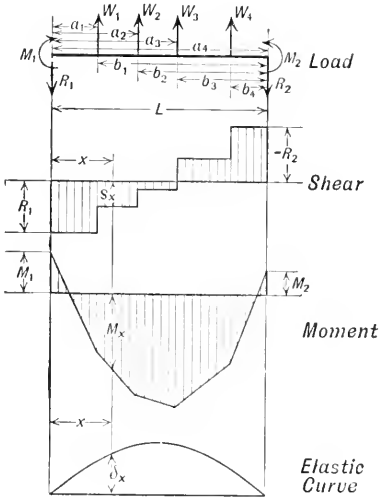
$$M_x = M_1 + S_{+1}x \qquad M_{x'} = M_1 + S_{+1}x' + W(x' - a)$$

Elastic curve $\delta = \dfrac{x(x-L)}{EI}\left[\dfrac{M_1}{2} + \dfrac{S_{+1}}{6}(x+L) - \dfrac{Wb^3}{6\,L(x-L)}\right]$

and $\delta' = \dfrac{x'(x'-L)}{EI}\left[\dfrac{M_1}{2} + \dfrac{S_{+1}}{6}(x'+L) - \dfrac{W}{6(x'-L)}\left(\dfrac{b^3}{L} - \dfrac{(x'-a)^3}{x'}\right)\right]$

$$EI\,\delta_a = -\frac{M_1ab}{2} - \frac{S_{+1}ab(L+a)}{6} - \frac{Wab^3}{6\,L}$$

Case 15. Restrained Beam with Several Concentrated Loads.



$$R_1 = \frac{M_2 - M_1}{L} - \sum_0^L \frac{Wb}{L} \qquad R_2 = \frac{M_1 - M_2}{L} - \sum_0^L \frac{Wa}{L}$$

$$S_x = S_{+1} + \sum_0^x W = \frac{M_2 - M_1}{L} - \sum_0^x \frac{Wb}{L} + \sum_x^L \frac{Wa}{L}$$

$$M_x = M_1 + S_{+1}x + \sum_0^x W(x-a) = M_1 + \frac{M_2 - M_1}{L}x - x\sum_0^L \frac{Wb}{L} + \sum_0^x W(x-a)$$

$$\delta = \frac{x(x-L)}{EI}\left[\frac{M_1}{2} - \frac{S_{+1}}{6}(x+L) - \sum_0^L \frac{Wb^3}{6\,L(x-L)} + \sum_0^x \frac{W(x-a)^3}{6\,xL(x-L)}\right]$$

13 : 2. The Method of Work — Any changes in the lengths of its elements will result in the deflection of a structure. The magnitude and direction of the deflection of any given point on the structure will depend entirely on the location of the point, the magnitude of the changes in length, and the dimensions of the structure. Therefore, if we can derive a general expression for the deflection of *any* point due to the change in length of *any* element we can determine, by summation, the total deflection of that point due to any system of length changes of the elements. The system of length changes may be caused by the application of loads to the structure, change in temperature of the elements, or anything else. The deflections are purely geometrical results of the length changes and the cause of a given system of such changes has no other influence on the resulting deflections.

The simplest type of structure for which a general expression can be written for the deflection of any point in terms of the change in length of any element is a simply supported truss like that shown in

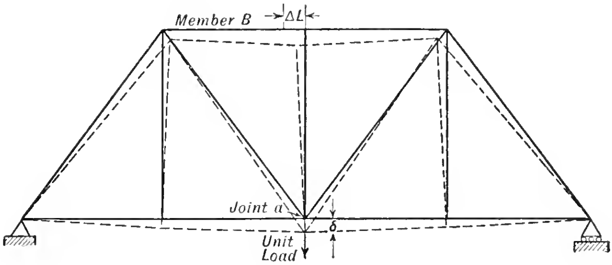


Fig. 13 : 1

Fig. 13 : 1. Suppose it is desired to find the deflection of joint a due to a change in length ΔL of member B. It would be possible but very difficult to develop a workable expression for the relationship between these quantities if we confined ourselves to consideration of the geometrical properties of the truss, so it is necessary to resort to some stratagem. Let us assume therefore that before member B suffered its change in length ΔL, that the truss had been subjected to a load of unity acting at joint a as shown in Fig. 13 : 1. This unit load would have caused each of the various truss members to be subjected to external loads at their ends, producing axial tension or compression in the members, and we will call the loads to which member B was subjected, s.

When member B was changed in length, joint a moved, and as that joint was the point of application of the unit load, that unit load did work equal to unity, the force, times the component of the deflection

of joint a parallel to the unit load, the distance. At the same time the ends of member B went through a relative movement, ΔL, so the forces, s, acting on that member also did work, equal to $s\Delta L$, which was stored up in that member as potential energy. If we assume the other members of the truss were rigid and suffered no change in length, the forces on them resulting from the external load of unity did no work, and the work done by member B represents all of the internal work resulting from the movement of the unit external load.

It is a consequence of the law of the conservation of energy that when the point of application of an external force on a body moves, the energy represented by the work done by that force as a result of this movement will all be absorbed by the body. If we assume, as we will in this case, that there is no acceleration imparted to the body as a whole, and the elastic limit of its material is not exceeded, this energy will be taken up as potential energy in the elastic elements of the body. The external work done and the internal work done are then independent measures of the energy transfer involved and must be numerically equal.

In the case in question, we can write $\delta = s\Delta L$ where δ is the deflection of the point of application of the unit load parallel to its line of action due to the change in length ΔL of member B. It must be noted that this expression gives the *component* of the deflection of joint a parallel to the line of action of the unit load assumed. If we had assumed the unit load to act in some other direction, the same reasoning would have held good, but the numerical value of s would have been different, and the expression would have given the component of the deflection of a parallel to this new direction of the unit load. In order, therefore, to determine completely the deflection of any truss joint due to the change in length of any given member it is necessary to apply the method at least twice. Usually this would be done by determining the vertical and horizontal components of the deflection separately, using first a vertical and then a horizontal unit load at the joint in question.

The expression just derived may be used to determine the deflection of any joint of a truss due to the change in length of any *one* member. Usually it is desired to determine the deflection of a joint due to all of the truss members suffering changes in length simultaneously. This quantity will be the sum of the deflection producing effects of the separate changes in length, and as B was "any member," the formula

$$\delta = \Sigma s\Delta L \qquad 13:1$$

can be written for the deflection of any given joint due to a system of length changes acting simultaneously.

If it is desired to determine the deflection of a truss joint due to the

action of external loads on the truss, the changes in length, ΔL, will be those due to the internal forces produced by the external loads. If for any truss member, P is the axial load caused by the external forces, A its cross-sectional area, L its length, and E the modulus of elasticity of its material, ΔL for that member will be PL/AE and the formula for deflection will be

$$\delta = \Sigma sPL/AE. \qquad 13:2$$

This method of computing deflections, being based on the equation of the external work and internal work done by a hypothetical system of forces acting simultaneously with the length changes causing the deflections, is called the Method of Work. The unit force at the point where the deflection is desired and the forces, s, caused by it are not a part of any force system actually causing deflections, but are purely a measuring device making it possible to write a simple algebraic expression for the complex geometrical relationship between the deflection of any point and the length changes which cause it. In developing formulas 13 : 1 and 13 : 2, it was tacitly assumed that the forces acting on the various truss members were not affected by the change in shape of the truss due to the deflections of its joints. This is an approximation but the error involved is small with most engineering structures and can be neglected.

In numerical computations, it is important to use the proper signs for the various quantities. If tension and increase in length are both indicated by the plus sign, a positive value of $s\Delta L$, sPL/AE, or summations of those quantities will indicate that the deflection is in the same direction as was assumed for the unit external load, and vice versa. That this must be the case can be determined from a study of a simple structure such as a single bar under an axial load. Proof of this proposition by the student would probably aid him greatly in understanding the method.

13 : 3. Numerical Example — Find the vertical and horizontal components of the deflection of joint $L5$ of the truss in Fig. 13 : 2 under the loading shown. The chords and end diagonals are $3/4 \times 0.035$ and the other members $5/8 \times 0.035$ steel tubes with a modulus of elasticity of 29,000,000 lb. per sq. in.

The preliminary steps are to compute for each member the values of L/A, its length divided by its sectional area, P, the axial load due to the system of external loads shown; S_v, the axial load due to an external load of unity acting up at joint $L5$; and S_h, the axial load due to an external load of unity acting to the right at joint $L5$. These values are listed in columns 2 to 5 of Table 13 : 2.

The values given in Columns 6 and 7 of the table are obtained by multiplying the proper quantities in the preceding columns.

From Table 13 : 2 we have $\sum S_v \dfrac{PL}{A} = -1{,}933{,}650$

Whence $$\delta_v = \sum S_v \frac{PL}{AE} = -\frac{1{,}933{,}650}{29{,}000{,}000} = -0.0668 \text{ in.}$$

Also $$\delta_h = \sum S_h \frac{PL}{AE} = +\frac{641{,}200}{29{,}000{,}000} = +0.0221 \text{ in.}$$

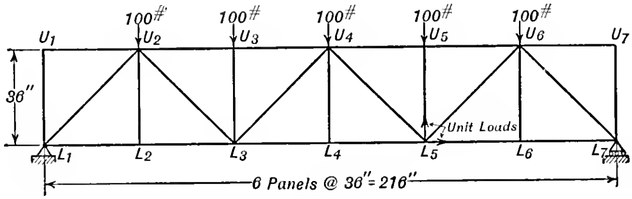


FIG. 13 : 2

Joint $L5$ will therefore deflect 0.0668 in. downward and 0.0221 in. to the right; the minus sign of δ_v indicating deflection in the direction opposite to that of the vertical unit load, and the positive sign of δ_h indicating deflection in the same direction as that of the horizontal unit load.

In making up Table 13 : 2 it should be noted that those members like $U1$–$U2$ for which P was zero were omitted as they obviously would not contribute to the deflection. The same thing could have been done with $U3$–$L3$ and $U5$–$L5$ for which both S_v and S_h are zero. The upper chord members could also have been grouped in pairs in which the values of P, S_v, and S_h and A were the same for both members of the pair, the length L being taken as the sum of the lengths of the members paired as was done for the lower chord.

Care must be taken in computations of this kind to use the proper signs as the work in members like $U4$–$L5$ in which s and P are of opposite signs must be subtracted from that done in the members in which s and P are of the same sign.

Where all members are of the same material, they will have the same value of E, and it is usually simpler to summate the values of $s\dfrac{PL}{A}$ and divide the sum by E as is done in this case than to compute the individual values of $s\dfrac{PL}{AE}$.

TABLE 13 : 2

COMPUTATION OF DEFLECTIONS OF JOINT *L5*

Member 1	P 2	S_v 3	S_h 4	$\frac{L}{A}$ 5	$S_v \frac{PL}{A}$ 6	$S_h \frac{PL}{A}$ 7
U2–U3	−400	+0.6667	0	458	−122,123	0
U3–U4	−400	+0.6667	0	458	−122,123	0
U4–U5	−400	+1.3333	0	458	−244,246	0
U5–U6	−400	+1.3333	0	458	−244,246	0
L1–L3	+250	−0.3333	+1.0	916	−76,333	+229,000
L3–L5	+450	−1.0000	+1.0	916	−412,200	+412,200
L5–L7	+250	−0.6667	0	916	−152,666	0
U3–L3	−100	0	0	555	0	0
U5–L5	−100	0	0	555	0	0
L1–U2	−353.55	+0.4709	0	649	−108,058	0
U2–L3	+212.13	−0.4709	0	786	−78,513	0
L3–U4	− 70.71	+0.4709	0	786	−26,171	0
U4–L5	− 70.71	−0.4709	0	786	+26,171	0
L5–U6	+212.13	−0.9418	0	786	−157,026	0
U6–L7	−353.55	+0.9418	0	649	−216,116	0
					−1,933,650	+641,200

13 : 4. Method of Work for Beams — The Method of Work can be applied to beams subjected to bending, and provides a useful method of determining the deflections of beams of non-uniform section. The fundamental theory is the same as for trusses, but the practical formulas differ. In both cases, the deflection is determined by equating the external and internal work done by a previously applied external load of unity and its induced internal forces when the changes of length of the elements of the structure occur to cause the deflection. In the case of the truss the problem was simplified by the fact that each truss member could be considered as of uniform cross-section and to be subjected to an axial load of constant magnitude along its entire length, and that the work done by the force s could easily be computed. In a beam subjected to bending, the axial loads on the fibers vary as the bending moment varies along its length, and vary also with the distance of the fiber from the neutral axis of the beam. The basic element for which the internal work must be computed must therefore be of infinitesimal cross-sectional area so that the distribution of stress may be assumed uniform, and of infinitesimal length so that the total load on its section may be assumed constant along its entire length. The element would then be, as shown in Fig. 13 : 3, that part of the beam bounded by two cross-sections, dx apart, two planes parallel to the neutral plane, dy apart, and the sides of the beam, b apart. If f' is the

unit stress on the element due to the unit external load at the point the deflection of which is being computed, and f the unit stress due to the bending loads causing deflection,

$$\delta = \frac{f'b\,dy\,f\,dx}{E}$$

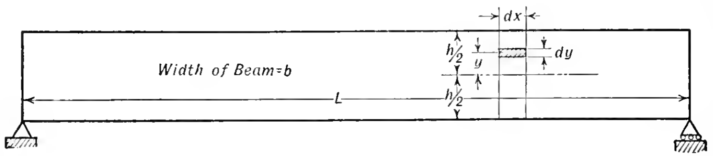


FIG. 13 : 3

As the cross-sectional area of the element is $b\,dy$, $f'b\,dy$ corresponds to the term s of the formula for trusses, and $f\,dx/E$ is the change in length of the element due to the bending loads. Therefore δ is the deflection parallel to the unit load of the point at which the unit load is applied, caused by the change in length of the one element considered. In order to determine the deflection due to the changes in length of all the elements in the beam, it is necessary to integrate with respect to y so as to take in the entire cross-section, and with respect to x so as to cover the entire length of the beam. If the limits of the cross-section be designated by $+\frac{h}{2}$ and $-\frac{h}{2}$ and the length of the beam is L, we can write as the expression for the deflection of any point due to the deformation of all the elements of the beam

$$\delta = \int_0^L \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{f'b\,dy\,f\,dx}{E}$$

If we now designate by m the bending moment at any section due to the unit external load and by M the bending moment due to the loads causing the deflection being computed,

$$f' = my/I \quad \text{and} \quad f = My/I$$

Then $$\delta = \int_0^L \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{mybMy\,dy\,dx}{I^2E} = \int_0^L \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{mMy^2b\,dy\,dx}{I^2E}$$

But $$\int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2b\,dy = I$$

So that $$\delta = \int_0^L \frac{mM\,dx}{EI}. \qquad 13:3$$

In the derivation of Formula 13 : 3, no account is taken of the deflection due to shear, and in this respect the formula fails to give the actual total deflection. In practically all cases in which the engineer is interested, however, the deflection due to shear is so small compared to that due to bending that the former is neglected, or is allowed for approximately by using a slightly reduced value for the modulus of elasticity.

The cases of the truss with loads acting only at the joints, and the beam subjected to bending only are the simplest ones to which the method of work can be applied. It can be used for more complex cases also by proper modification and combination of the formulas for the two cases discussed. An example of the use of the formula for a beam subjected to bending is given below.

13 : 5. Numerical Example — Find the vertical deflection of the free end of a horizontal cantilever beam of length L and moment of inertia I due to a down load of w pounds per inch as shown in Fig. 13 : 4.

Assume a unit up load to act at the free end. Then $m = x$.

From the down load w, we have $M = -wx^2/2$

$$\text{Then } \delta = \int_0^L \frac{mM\,dx}{EI} = \int_0^L \frac{x(-wx^2)\,dx}{2\,EI} = \frac{-w}{2\,EI}\int_0^L x^3\,dx$$

$$\delta = \frac{-w}{2\,EI}\cdot\frac{x^4}{4}\Big]_0^L = -\frac{wL^4}{8\,EI}$$

This is the same formula as was obtained for this case by double integration and given in Art. 13 : 1. The minus sign indicates that the direction of the deflection is opposite to that of the unit load, which is obviously correct for this case.

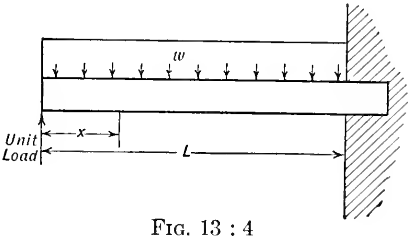


FIG. 13 : 4

In the above example the load per inch and the moment of inertia of the beam were both assumed constant, greatly simplifying the numerical work. If these quantities had varied it would have been necessary to express them in terms of x and to have integrated the resulting expression which would probably be much more complex. If no single algebraic expression can be written for $mM\,dx/EI$ which will apply to the entire length of the beam it is necessary to divide the beam into sections for each of which such an expression can be written, integrate each between the limiting values of x to which it applies, and add the results. To allow for this possibility it might be better to write the formula as

$$\delta = \sum \int \frac{mM\,dx}{EI} \qquad 13 : 4$$

13 : 6. The Principle of Reciprocal Deflections — If a truss be subjected to a load W at some joint a, the deflection of any other joint b in any given direction will be $\Sigma sPL/AE$ in which s is the load on any member due to a unit load at b in the direction of the desired deflection and P the load in that member due to W. This expression could also be written $\Sigma srWL/AE$ if we designate by r the load in the member due to a unit load at a in the same direction as W. Similarly the deflection of point a in the direction assumed for W in the first case, due to a load of W acting at b in the direction of the unit load there in the first case, will be $\Sigma srWL/AE$. The same reasoning is equally valid for beams using the formula $\int mM\,dx/EI$. Thus it is possible to prove Maxwell's Law of Reciprocal Deflections: The deflection of any point a of a structure due to a load W, acting at any other point b on that structure, is equal to the deflection of b due to a load W acting at a, provided the load at and deflection of each point are in the same directions for the two cases. This principle is extremely useful in many deflection computations and should be thoroughly understood.

A good example of the application of the principle of Reciprocal Deflections is found in the computations of the deflection of the N-strut point on the spar of an internally braced biplane. In order to design an internally braced biplane properly it is necessary to compute the deflection of the point on the spar at which the N-strut for equalizing the deflections of the wing tips is connected, under the distributed loads coming on the spar in each of the four standard loading conditions for the wings, and also under various concentrated loads applied from the N-strut.

The first step is to compute the ordinates of a curve of spar deflections under a concentrated load of unity at the N-strut point, this curve being drawn for the entire length of the spar, including the tip section out-board of the N-strut where it will be a straight line. As the deflection of any point on the spar will vary directly with the magnitude of the concentrated load at the N-strut point, the deflections due to any concentrated load at that point can be obtained by simple multiplication.

By the law of reciprocal deflections, each ordinate to the deflection curve will represent not only the deflection of the corresponding point on the spar due to a unit load at the N-strut, but also the deflection of the N-strut point due to a unit load acting at the point on the spar corresponding to that ordinate. In other words the curve of deflections under a unit load at the N-strut point is also an influence line for the deflection of the N-strut point. Treating the original curve from the

latter point of view, it is a simple matter to draw a new curve, each ordinate of which is the product of the corresponding ordinates of this influence line and the curve of distributed loading, and the area under the last drawn curve will represent the deflection of the N-strut point under the distributed loading in question. This method is particularly useful when the deflections of a single point under several geometrically dissimilar loadings are desired. In using it not only the method of work may be used to compute the deflections under the unit load, but any of the other methods described in this chapter may be employed.

The principle of reciprocal deflections may often be used to advantage in the determination of the magnitude of a third reaction on a beam or truss under a simple or complex loading. For example, let us assume a unit load to be applied at point b, the proposed point of support to be added near the middle of a simple beam, and let the deflection at b due to that load alone be 2 in., the deflection at some point a be 1/4 in. Then the deflection at b would be 1/4 in. if the unit load were applied at a and a reaction of $1/4 \div 2$, or 1/8 of a unit load, applied at point b would serve as a reaction and prevent the deflection of b under the load at a. Therefore, since a may represent any point on the beam or truss, we note that we may obtain an influence line for the reaction at b by dividing the ordinates to the curve of deflections obtained with the unit load at b by the deflection computed for point b itself.

This principle is used in all of the methods of mechanical analysis, such as the Beggs' Method, and is of great assistance in the solution of stresses in members of trusses which are indeterminate as regards the outer forces or reactions.

13 : 7. Deflection of Beams by Method of Moment Areas — The deflection of any point b of a beam measured from a line tangent to the elastic curve of the beam at any other point a is equal to the area under the portion of the M/EI curve between a and b multiplied by the distance from the centroid of that area to point b. This proposition, the application of which is the Method of Moment Areas for computing beam deflections, can be proved as a corollary of the basic formulas of either the Method of Successive Integration or that of Work.

As the proof of the validity of this method and its application to cases involving simple shapes of curves of M/EI are covered by the texts on Mechanics of Materials, those subjects will not be discussed here. The Method of Moment Areas, however, is specially well suited for use in cases where the M/EI curve is irregular, due to an irregular loading curve or variation in the moment of inertia of the beam, and its application to such cases will be described.

The simplest case for the application of the Method of Moment

Areas is that of a cantilever fixed at one end and for which the deflections from a line tangent to the elastic curve at the fixed end are desired. For this case it is necessary only to draw a curve, each ordinate of which is equal to the area under that part of the M/EI curve between the point on the beam for which the ordinate is drawn and the fixed end, multiplied by the distance from that point on the beam to the centroid of that part of the area under the M/EI curve. This process could also be described as drawing a bending moment curve for a hypothetical cantilever beam, supported at the free end of the actual beam and subjected to a loading numerically equal to the ordinates of the M/EI curve at each point.

For a beam with two supports, either a beam simply supported at its ends or a beam with one or two overhanging ends, the process is more involved as the deflections desired are normally those from a line joining the two supports, and in general this line will not be a tangent to the elastic curve. In this case it is necessary to compute the deflections of both the desired reference line and the elastic curve of the beam from some tangent to the elastic curve and obtain the desired deflections by subtraction. Since the computations for the beam with two supports include those for a cantilever as an intermediate step, the illustrative example given below is for the more complex case. Also, as either graphical or analytical methods of computation may be used and both have their advantages, the illustrative example is worked out by both methods.

13 : 8. Numerical Example — Graphical Method — Consider Fig. 13 : 5 to be a diagram, correct as to scale, of the M/EI curve for a beam simply supported at its ends, and assume that it is desired to determine the deflection of this beam from a line joining the supports. The sources of the combination of loading and shape of beam that result in the shape of M/EI curve shown will not be gone into as they have no bearing on the method of computing deflections.

The first step is to determine the deflections of the elastic curve from some tangent to it. For convenience the tangent at the left hand support will be used, though any other tangent could be employed. As indicated in Art. 13 : 7 the process of determining the deflections from a tangent to the elastic curve at the left support is the same as that of drawing the curve of bending moments on a hypothetical cantilever supported at the right end of the actual beam, and subjected to a loading represented by the M/EI curve of Fig. 13 : 5. This step is carried out by the method described in Art. 8 : 5, though the details vary somewhat from those of the illustrative example of that article because the loading on the hypothetical beam is distributed instead

of being a group of concentrated loads, and the hypothetical beam is a cantilever instead of being supported at two points.

The first difference in procedure is in the method of drawing the force polygon of Fig. 13 : 6. Instead of being able to lay off the loads on the beam directly, it is necessary to divide the beam into short sections and assume that each section is subjected to a single concentrated load equivalent to the distributed load. This can be done by drawing vertical lines at the division points or "stations" between the sections, as shown in Fig. 13 : 5, dividing the area under the loading

Fig. 13 : 5

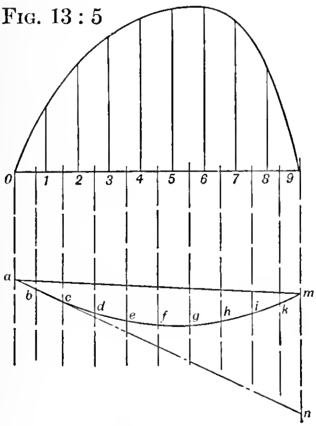


Fig. 13 : 7

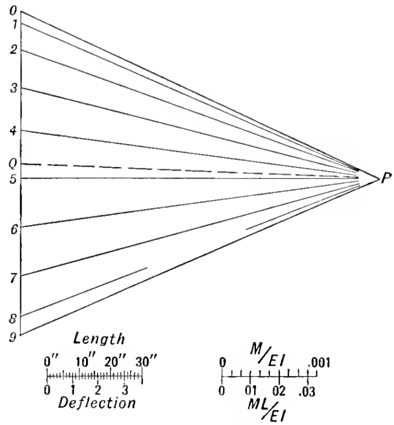


Fig. 13 : 6

curve into segments. The area of each segment then represents the magnitude of the load on the corresponding section of the beam, and its centroid a point on the line of action of that load. It is usually desirable to divide the beam into sections of equal length, and so short that the centroid of the load on the section can be assumed to act at the middle of its length without serious error. Sometimes, as in the cases of the end segments of Fig. 13 : 5, it is better to assume the loading to be triangular with the centroid of the segment at a third point of the length of the section. The loading curve having been divided into segments, the force polygon can be drawn as in Fig. 13 : 6. In this figure the line 0–1 represents to scale the area of the segment of the loading curve of Fig. 13 : 5 on the section 0–1 of the beam. Similarly 1–2 of the force polygon represents to scale the area of the segment 1–2 of the loading curve, and so on for the remainder of the segments of the loading curve.

The string polygon, called in Fig. 13 : 7 the "deflection diagram," is started at b which is any point on the line of action of the resultant

load on the left-hand section of the beam, 0–1. The line a–b–n is drawn parallel to the ray 0–P of the force polygon, and b–c parallel to the ray 1–P. From c the intersection of b–c and the line of action of the resultant load on the section 1–2, c–d is drawn parallel to the ray 2–P. This process is continued until the line k–m is drawn, m being on a vertical through the right support of the beam.

As the beam is assumed to be a cantilever, no closing line is needed as yet, and the figure drawn may be considered either as a curve of bending moments for a cantilever beam supported at the right-hand end, under the distributed loading in pounds per unit of length indicated by the curve of Fig. 13 : 5, or as a plot of the elastic curve of a beam with the M/EI curve shown in that figure. If considered as a curve of bending moments the intercept between the straight line a–n and the broken line a–b–c– · · · –m on a vertical through any point on the beam in Fig. 13 : 5 represents the bending moment at that point. If considered as a plot of the elastic curve, the same distance represents the deflection of the point from the tangent to the elastic curve at the left support.

Fig. 13 : 7 does not give a true bending moment or elastic curve as concentrated loads have been substituted for the distributed load represented by the M/EI curve. The error can, however, be decreased by shortening the lengths of the sections into which the beam is divided, and the same result can be obtained by inscribing a smooth curve tangent to each of the parts of the broken line a–b–c– · · · –m.

So far the construction gives only the deflections from the tangent at one end of the beam, and the deflection of the desired reference line joining the supports must be determined. This can be done very easily in the graphical method as the broken line a–b–c– · · · –m is a plot to scale of the elastic curve of the beam. Since a represents the position of one support and m that of the other support, if the straight closing line a–m is drawn, the deflection of any point from a line joining the supports will be represented by the intercept on a vertical through the point between the straight line a–m and the broken line a–b–c– · · · –m.

In the above example all values of M and therefore of M/EI are assumed to be of the same sign. If the sign of M changes it is necessary to assume a point of division between sections of the beam at each point of inflection, and to lay off distances on the force polygon in one direction for positive areas under the M/EI curve and in the other direction for negative values. The same is true when using the graphical method for computing bending moments if the load changes in direction.

It should be noted that the funicular polygon of Fig. 13 : 7 repre-

sents not only the shape of the elastic curve for a beam loaded in such a way that the variation in M/EI is as shown in Fig. 13 : 5, but that mathematically it is the second integral of the curve of M/EI without regard to the constants of integration. They in turn are introduced when the axis $a-m$, the base line from which the final deflections are measured, is drawn so that it passes through a and m, the two points at which the deflections are zero. It follows, therefore, that if we have a simple span at one or both ends of which there are cantilever overhangs, the deflections may be found by plotting the M/EI curve for the entire length of the beam, having due regard for the signs of the moments. A funicular ploygon may then be constructed for this curve without regard to the points of actual support for the beam and this polygon will give the shape of the elastic curve for the given beam under the applied load. If the deflections at the actual points of support are zero, the axis from which the deflections are to be measured may be drawn through those two points and ordinates lying on one side of the axis will represent positive deflections, ordinates on the other side, negative.

The scale of deflections in Fig. 13 : 7 must be determined. Assume the linear scale of Fig. 13 : 5 to be 1 in. = 30 in. (All of the scales assumed in this discussion are for the original diagram which has been reduced to about half size in printing.) Assume also that the scale of M/EI is 1 in. = 0.001 in.$^{-1}$, the dimensional quality of M/EI being inches to the minus one power. One square inch of Fig. 13 : 5 will then represent 0.03 units of the non-dimensional quantity ML/EI or "elastic load."

In Fig. 13 : 6 let 1 in. = 10/9 sq. in. of Fig. 13 : 5 or 0.0333 units of elastic load. The pole distance is 3.75 inches. Therefore on the deflection diagram of Fig. 13 : 7, one inch measured vertically represents a deflection of $30 \times 3.75 \times 0.0333 = 3.75$ inches. The deflection of point 5 of the beam as measured on Fig. 13 : 7 is 0.4 in. which represents an actual deflection of the beam of $0.4 \times 3.75 = 1.50$ inches.

13 : 9. Numerical Example — Analytical Method — The steps of the analytical method of moment areas for the computation of deflections are parallel to those of the graphical method. Again the first step is to compute the deflections from a tangent at one end by assuming the beam to be a cantilever fixed at the other end and computing the bending moments due to a distributed loading numerically equal to the M/EI curve of the actual beam. These computations are carried out for the beam of Fig. 13 : 5 by the method described in Art. 3 : 8 and are tabulated in the first ten columns of Table 13 : 3.

In Table 13 : 3 the column headings of the first ten columns are

like those which would be used in a bending moment computation except that quotation marks are used to indicate that instead of being a computation of true bending moments from loads, it is one of deflections from a curve of M/EI. The values of "w" in column 2 are the values of M/EI scaled from the curve of Fig. 13 : 5 multiplied by 1,000. This multiplier was used to avoid the necessity of using many numbers smaller than unity in the computations. The values in column 3 are the sums of the adjacent values of column 2. The values of "W" in column 5 give the magnitudes of the areas of the respective segments of the M/EI curve, and represent the concentrated loads used in the graphical computation. They are equal to half the section length multiplied by the sum of the ordinates of the M/EI curve at the ends of the segments. The same values could have been obtained by multiplying the total length of the section by the average ordinate as was done in the illustrative example of Art. 3 : 8. The method used here, however, is sometimes easier to apply though its justification may not be so easily seen.

TABLE 13 : 3

COMPUTATION OF BEAM DEFLECTIONS

1 Sta.	2 "w"	3 Sum	4 L	5 "W"	6 Arm	7 "M_w"	8 "S"	9 "M_s"	10 "M_c"	11 δ_t	12 δ
0	0						0		0	0	0
		0.69	10	3.45	3.33	11.50		0			
1	0.69						3.45		11.50	0.51267	0.50117
		1.81	10	9.05	4.61	41.72		34.50			
2	1.12						12.50		87.72	1.02533	0.93761
		2.54	10	12.70	4.80	60.96		125.00			
3	1.42						25.20		273.68	1.53800	1.26432
		3.02	10	15.10	4.90	73.99		252.00			
4	1.60						40.30		599.67	2.05066	1.45099
		3.31	10	16.55	4.94	81.76		403.00			
5	1.71						56.85		1084.43	2.56333	1.47890
		3.44	10	17.20	4.99	85.83		568.50			
6	1.73						74.05		1738.76	3.07600	1.33724
		3.31	10	16.55	5.08	84.07		740.50			
7	1.58						90.60		2563.33	3.58866	1.02533
		2.65	10	13.25	5.32	70.49		906.00			
8	1.07						103.85		3539.82	4.10133	0.56151
		1.07	10	5.35	6.67	35.67		1038.50			
9	0						109.20		4613.99	4.61399	0

The values in column 6 are the distances from the right hand end of each segment to its centroid assuming the segment to be a trapezoid. Column 7 lists the values of "M_w" the moment of the load on each section about the right end of that section. "M_w" corresponds to M

of Table 3 : 1 in the illustrative example in Art. 3 : 8. "S" of column 8 is the total area under the M/EI curve to the left of the various stations on the beam or the "shear" on the hypothetical cantilever. "M_s" is the moment of the "shear" at each station about the next station to the right and corresponds to M'' of Table 3 : 1. "M_c" is the cumulative sum at each station of the preceding values of "M_w," "M_s," and "M_c" obtained in the same manner as the values in the last column of Table 3 : 1. This value may be assumed to be either the bending moment on a hypothetical cantilever beam or one thousand times the deflection of the beam from a tangent to the elastic curve at the left hand support. The factor of one thousand must be inserted here to neutralize its effect when used in the computation of the values of "w."

It may be noticed that the values of "M_c" are so large in comparison to those of "M_w" that no appreciable error would have resulted if all of the values in column 6 had been taken as 5.00 or one-half the section length. This is usually the case in practical deflection computations as well as in those of bending moments and the simplifying assumption is allowable unless special precision of the results is attainable and desired.

Column 10 of Table 13 : 3 gives the deflection of the right support of the beam from a line tangent to the elastic curve at the left support. $\delta_r = 4.61399$ in. corresponding to the distance mn of Fig. 13 : 7. The distance at any station between the tangent to the elastic curve at the left support and the line joining the two supports will evidently be $\delta_r x/L$ where δ_r is the distance between those lines at the right support, L the distance between the two supports, and x the distance from the left support to the point in question. These distances are shown in column 11 and the deflections desired in column 12. In the case of a cantilever beam the last two columns would be omitted as the deflections desired would be those from a tangent to the curve at the fixed end of the cantilever and the moments of the areas under the M/EI curve could be obtained so as to get those values directly.

It is of interest to note the agreement between the results of the two methods of computation as applied to the same problem. From the graphical construction the deflection of Station 5 was 1.50 in., while the same value from the analytical computation was 1.48 in. The error is about two per cent and would have undoubtedly been smaller if larger scales had been used in the graphical work. In fact the deflection as measured was a scant 0.4 inches showing that if it could have been measured more precisely, the apparent difference between the results of the two methods would have been smaller.

It may be noted that the computations by the graphical and analytical

methods are logically identical, the only difference being in the manner in which the additions, multiplications, etc., are made. In any given case some of the stages could be done graphically and the others analytically without changing the results. In both cases, when the beam is homogeneous, the modulus of elasticity E is a constant and the numerical work can often by simplified by using a curve of M/I instead of M/EI, using the method to obtain values of $E\delta$ which would be divided by E to obtain the true values of deflection, δ. This practice is often advisable in order to work with whole numbers of reasonable size instead of small decimals. The same result can be obtained by assuming a fictitious value of E, such as 29 lbs. per sq. in. for steel instead of the true value of 29,000,000 lbs. per sq. in., and dividing the final results by the proper factor to obtain the correct results, in this case 1,000,000 or the similar device used above can be employed. In the case of a beam of uniform cross-section, the same principle can be applied, using the curve of M instead of that of M/EI in most of the computations and dividing the results by EI to get the actual deflections in inches.

13 : 10. Deflection of Trusses by Method of Elastic Weights — The Method of Work as applied to trusses suffers from the defect that a separate computation involving all the members of the truss must be gone through for each position of the unit load to determine the deflection of each joint. When the deflections of several joints are desired the numerical work becomes very tedious even though many of the figures can be used several times. It is possible however to determine "elastic weights" proportional in magnitude to the changes in length of the individual truss members and so located that the curve of bending moments on a beam subjected to these weights will be numerically equal to the deflections of the joints of the truss.

The basic formula for the deflection of any joint of a truss due to the change in length of any one member is $\delta = s\Delta L$. If, therefore, we should draw an influence line for s in the member in question, the unit load being assumed to travel along one chord of the truss, the ordinates of this influence line at the various joints along that chord multiplied by ΔL would give the deflections of those joints due to the change in length of the member. If such influence lines were drawn for all of the members of the truss, the sums of the ordinates at any joint to all of these influence lines, each multiplied by the proper value of ΔL, would give the deflection of that joint. This in itself would not be any better than the method of work, but advantage can be taken of the fact that in all common forms of trusses, the influence lines would be of the same shape as the moment curves of certain simple load

systems. Instead of getting all of these influence lines separately, therefore, the practice is to compute the loads whose moment curves are equivalent to the separate influence lines and then compute the bending moment curve for the entire system of loads in a single operation and thus obtain the deflection desired. In order to simplify the proof and description of the application of this method, it is advisable to define certain terms that will be used. As deflections of the joints on only one chord of a truss can be obtained in a single application of this method, that chord will be called the *main chord* of the truss. If it is desired to obtain the deflections of the joints along both chords, the method must be applied twice, first with one chord and then with the other as the main chord. The hypothetical beam on which the elastic weights are assumed to act will be called the *phantom beam*. The *index section* of the truss for a member is the section by which the truss is assumed to be cut when determining the load in that member by the method of moments. The *moment center* of a member is the point about which moments would normally be taken when computing the load on the member by the method of moments, i.e., the intersection of the other two members cut by the index section. The *secondary moment center* of a web member is the distant end of the main chord member cut by the index section of that member. The *moment arm* and *secondary moment arm* are the perpendicular distances from the member to its moment center and secondary moment center respectively.

Suppose it is desired to determine the elastic weights of the members of a simply supported truss like that shown in Fig. 13 : 8 in order to compute the deflections of the joints along the bottom chord. The bottom chord will then be the main chord for this computation. The first step is to determine the shape and ordinates of the influence lines for the various members, the unit load being assumed to move over that main chord from m to n. Consider first the chord member bc. The influence line for this member will be as shown in Fig. 13 : 9, a triangle with its maximum ordinate $\dfrac{mn \cdot hn}{ch \cdot mn}$ at h, the moment center of member bc, the positive ordinate indicating that if the unit load acts up, member bc will be subjected to tension. This influence line, however, is numerically identical with the bending moment curve for a phantom beam supported at m and n and subjected to a downward or negative load of $1/ch$ acting at h. Thus the moment curve of the load of $-1/ch$ on the phantom beam is equivalent to the influence line for s in member bc, and the moment curve for a load of $-\Delta L/ch$ acting at h would give the deflections of the joints of the main chord due to change in length ΔL of member bc. As ch is the moment arm of member bc,

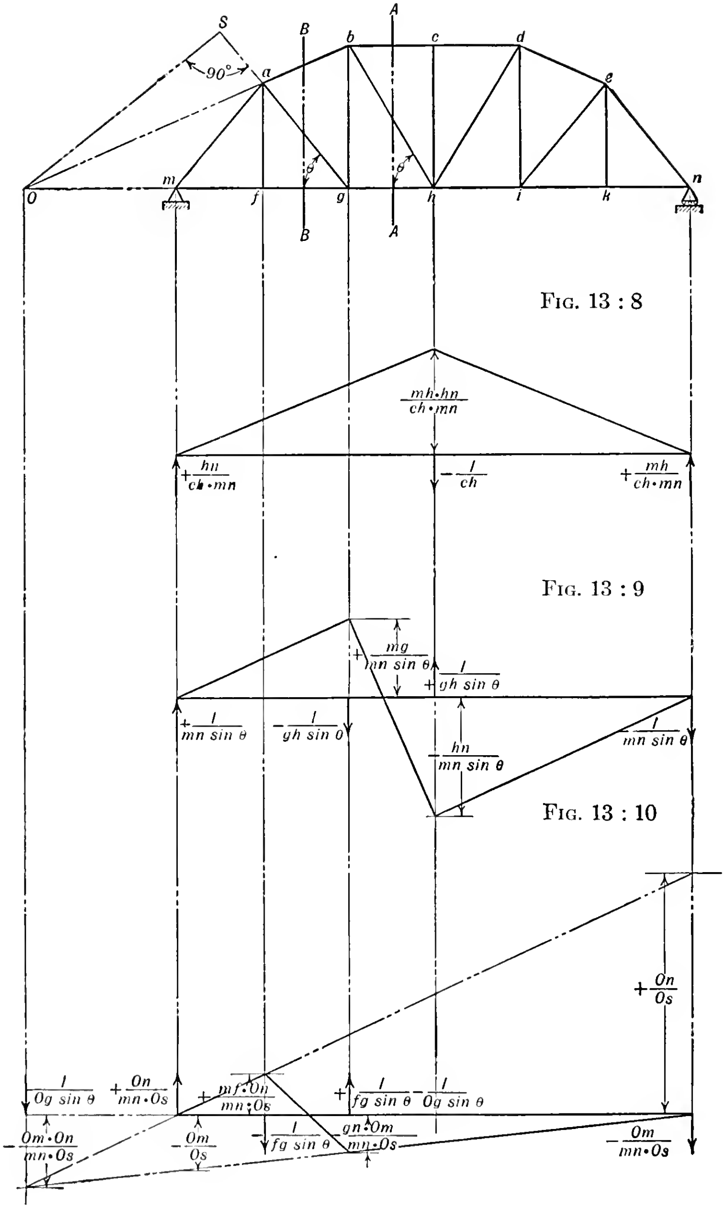


FIG. 13 : 8

FIG. 13 : 9

FIG. 13 : 10

FIG. 13 : 11

we can say that the elastic weight of that member is a load at its moment center equal to its change in length divided by its moment arm. Since it can be shown that this statement will apply to all of the chord members, whether parallel to the main chord or at an angle to it, we obtain the *chord member rule;* the elastic weight of any chord member of a simply supported truss is a load numerically equal to the change in length of that member divided by its moment arm, and applied at its moment center to a phantom beam supported at the points of support of the actual truss.

If member bc is subjected to tension, ΔL will be positive, the deflections will be of the same sign as the ordinates of the influence line for s, but the bending moment on A–A the index section for bc, as indicated by the tension in bc and the sign of the elastic weight of bc will be negative. If bc were in compression, ΔL would be negative, positive bending moment on the index section would be indicated, and a positive elastic weight would be needed to obtain bending moments of the same sign as the deflections. From this we can obtain the general rule that the sign of the elastic weight of a chord member and that of the bending moment on its index section, as indicated by the character of the load in the member, will be the same.

If a chord member is horizontal, its moment arm, R, will be equal to the vertical distance to the moment center. If the member makes an angle θ with the horizontal we will have $R = H \sin\theta = V \cos\theta$, where H is the horizontal distance to the member from its moment center and V the vertical distance. Either expression may be used to compute R in any given case, but usually $H \sin\theta$ is the more convenient.

Web members must be divided into two classes for the purpose of computing their elastic weights: members whose index sections cut parallel chords and those whose index sections cut non-parallel chords. The derivation of the rule for the former class is the simpler, and will be taken up first, using member bh of the truss in Fig. 13 : 8 as an example. The influence line for load in this member is shown in Fig. 13 : 10. As this influence line is numerically equivalent to the moment curve for a phantom beam supported at the same points as the actual truss and subjected to a counter-clockwise moment composed of forces $1/gh \cdot \sin\theta$, we can use the same reasoning as above and say that the elastic weight of member bh is composed of a down load at g and an up load at h, each numerically equal to $\Delta L/gh \cdot \sin\theta$. If instead of g and h we speak of the secondary moment center and the intersection of the web member with the main chord and note that $gh \cdot \sin\theta$ is its secondary moment arm, we can state the general *web member rule;* the elastic weight of a web member whose index section cuts parallel

chords is composed of equal and opposite loads forming a couple, applied at the secondary moment center of the member and its intersection with the main chord, each equal to the change in length of the member divided by its secondary moment arm. If the web member is vertical, θ becomes 90 degrees, and $\sin\theta = 1$, and the rule still holds.

The rule for the directions of the forces forming the elastic weight of a web member is that they form a clockwise couple if the shear on the index section, as indicated by the character of the load in the member, is negative, and vice versa. The truth of this rule can be determined most easily by working out the four possible cases depending on the sign of the load in the member and whether the portion of it to the left of the index section is above or below that section.

The influence line for load in a web member whose index section cuts non-parallel chords is of the type of the influence line for member ag as shown in Fig. 13 : 11. This influence line is numerically equivalent to that of bending moment for a phantom beam supported at m and n and subjected to a down load of $-1/fg \cdot \sin\theta$ at f and an up load of $1/fg \cdot \sin\theta - 1/0g \cdot \sin\theta$ at g. These two loads may be considered as the parallel components at f and g of a single load of $-1/0g \cdot \sin\theta$ acting at 0. Since $0g \cdot \sin\theta$ is the moment arm, R, we may state that the elastic weight of a web member whose index section cuts non-parallel chords can be obtained from the chord member rule except that, instead of applying it to the phantom beam at the moment center of the member, it is applied as two parallel components, one at the secondary moment center and the other at the intersection of the member with the main chord. Since $fg \cdot \sin\theta$ is the secondary moment arm of member ag, it can be seen that the elastic weight of member ag is the couple that would have been obtained from the web member rule modified by a load at g, its intersection with the main chord, numerically equal to the elastic weight according to the chord member rule. Study of the various cases encountered in practice shows that the amount of the couple obtained by application of the web member rule should be clockwise in accordance with the rule for web members between parallel chords. The load equal to the elastic weight as obtained from the chord member rule should act in such direction that the net load nearest the moment center of the member will be numerically the larger. If the shear on the index section as indicated by the load in the web member is positive and the moment center is to the left, this part of the elastic weight will be a negative or down load and vice versa. If the moment center is to the right of the member this rule should be reversed.

In the case of a cantilever truss fixed at one end, it can be shown that the rules derived above for a simply supported truss will apply

for the computation of the elastic weights, but that the phantom beam must be considered to be a cantilever with a fixed end at the free end of the actual truss.

When a truss is simply supported with a cantilever overhang, the deflections of joints between the supports can be computed as if the truss were simply supported without cantilever overhangs. In computing the deflections of the cantilever ends, the net reaction on the phantom beams for the inner span reversed in direction may be applied as an elastic load to the free end of the phantom beam for the adjacent cantilever to make proper allowance for the slope of the truss at that point.

A somewhat simpler method for dealing with a truss having a cantilever overhang is to extend the method described in Art. 13 : 8 where the funicular polygon is stated to represent the shape of the elastic curve. When using elastic weights we employ the bending moment diagram instead of the funicular polygon, but in either case the resulting curve represents the second integral of the "loading" curve. Hence, after having determined the changes in length of the members in the truss due to the applied loads, the points of support being considered in their actual locations, we may determine the elastic loads and apply them to a phantom beam having its supports at the extreme ends of the actual beam. The curve of bending moments obtained for the phantom beam under elastic loads applied in this way gives the shape of the elastic curve for the truss. An axis drawn through the points of zero deflection on the actual truss is then a base line from which to measure ordinates to the elastic curve in order to obtain the deflection of the truss. Care must, of course, be taken with the signs of the elastic loads. For chord members, they should be applied to the phantom beam, assumed to be supported at the extreme ends of the actual truss, so that they will produce moments at the respective index sections of the same character as is indicated by the actual change in length of the actual truss members. That is, if a top chord member in the actual cantilever is increased in length by the actual load, indicating negative bending moment, the elastic load for that member must be applied to the phantom beam so that it will produce negative moment on the phantom beam at the point where it is applied. For this case it would act upwards on the phantom beam. For web members the elastic loads should be applied to the phantom beam so that they will produce shear on the phantom, at the section corresponding with the index section of the member, of the same character as that indicated by the change in length of the actual member.

Limitations of space make it impossible to give the proofs of these

cases or even the complete proofs of the rules for the simply supported span, but they are given in I. C. 620.[1] It is believed, however, that enough has been given to permit the student to prove the truth of these rules for any special case in which he may be interested, and he is advised to prove a few of these cases in order to become thoroughly familiar with this method of computing deflections.

In cases where secondary web members are used in sub-divided trusses and others to which the rules above do not apply, it is necessary to determine the elastic weights in the same manner as above by determining the shape of the influence line for stress in the member in question and finding the load on the phantom beam whose moment curve is numerically equal to ΔL multiplied by the ordinates to this influence line.

13 : 11. Numerical Examples — The Methods of Elastic Weights and Work are illustrated in three numerical examples. These examples cover three types of support, the simply supported truss span, the cantilever with fixed support, and the simple span with an overhang. They also cover all of the types of members likely to be encountered in airplane work. In order to illustrate as many varying situations as possible no attempt is made to have the trusses practical, but as many types of construction were put into each truss as seemed desirable. The computations by the Method of Work are made for only one or two points on each truss, and are primarily to show the agreement between the two methods. In order to demonstrate that this agreement is exact and not just approximate, in the first two examples care has been taken in laying out the trusses, and selecting the values for the sectional areas of the members so that there will be no apparent error due to dropping of significant figures. Also, more significant figures have been used than would be desirable in practical computations, and in several places quantities are listed as fractions instead of using decimal notation in order to obtain this exact check. Though these devices were resorted to in order to obtain an exact check between parallel computations, they would not be necessary in practical work. The arrangement of the computations of elastic weights, the drawing of the sketch of the phantom beam showing the loads on it, and the method shown for presenting the computations of moments on the phantom beam are recommended for use unless the computer can devise a clearer and simpler method of presentation for his computations.

The first example covers the computation of the deflections of the lower chord joints of the truss shown in Fig. 13 : 12, by elastic weights

[1] "The Computation of Truss Deflections by the Method of Elastic Weights." U. S. Army Air Corps Information Circular, No. 620.

and a check of the vertical deflection of joint H by the Method of Work. The upper part of the figure shows the dimensions, the external loads, the axial load in each member due to those loads and, in parentheses, the load in each member due to a unit up load at joint H. Table 13 : 4 shows the computation of the elastic weights of the various members and the deflection of joint H by the Method of Work.

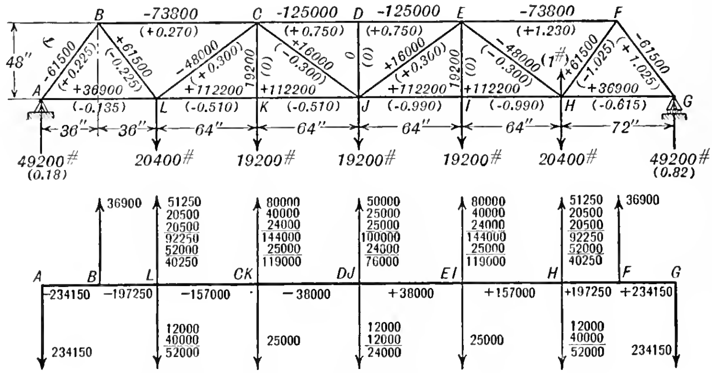
Column 1 gives the designation of the member, column 2, the axial load in thousands of pounds, this unit being used to conserve space. Column 3 gives the length of each member in inches and 4 its cross-sectional area in square inches. Column 5 gives the value of ΔL or rather of $E\Delta L/1000$. This value is equal to 0.001 PL/A. Column 6 gives the moment center of each chord member and 7 its moment arm, R. As the chords are parallel none of the web members have these values. Columns 8 to 11 respectively give the location of the secondary moment center, horizontal distance, h, from that center to the member, the sine of θ, the angle between the member and the horizontal, and r the secondary moment arm. Column 12 gives the elastic weights computed by the chord member rule, the individual values being the quantities in column 5 divided by those in 7. Column 13 gives the elastic weights obtained by the web member rule computed by dividing the quantities in column 5 by those in 11. Column 14 shows whether the single force elastic weights are up or down by the plus or minus sign, and whether the couples are clockwise or counter-clockwise by a self-explanatory symbol. Column 15 shows the points of application on the phantom beam of the various elastic weights, one point being shown for chord and two for web members. Column 16 lists the loads in the members due to the unit up load at joint H, and 17 the product of that value and the one in column 5. The algebraic sum of the values in column 17 is numerically equal to the deflection of joint H as determined by the method of work multiplied by 0.001 E.

It may be noticed that in Table 13 : 4 several pairs of members are grouped, as CD and DE. This is done only where the values of P and A are the same for both members and both have the same moment center. The length given is that for the pair. Member DJ is included in the table though it might have been omitted after inspection as it carries no load and its elastic weight is obviously zero.

The elastic weights of CK and EL cannot be obtained by the ordinary rules so their values are written in the table without reference to the columns. The influence line for load in CK is a triangle with its base extending from L to J and its apex with an ordinate of -1 at K. This is numerically equivalent to the curve of bending moment due to an up or positive load of $2/LK$, in this case 1/32 at K held in equilibrium

by loads of half that magnitude at L and J. The elastic weight of that member is therefore $+768/64 = 24$ at K and -12 at L and J. The elastic weight of EI is found in the same manner. If the loads in these members had been compressive (negative) instead of tensile (positive) the elastic weights would have had the same magnitudes but have acted in the opposite direction.

The lower part of Fig. 13 : 12 is a diagram of the phantom beam and shows the elastic loads at the different joints. The data shown include



$$
\begin{aligned}
-234{,}150 \times 36 &= -\ \ 8{,}429{,}400 \\
-197{,}250 \times 36 &= -\ \ 7{,}101{,}000 \\
&\ \overline{-15{,}530{,}400} = E\delta \text{ at } L \text{ and } H \\
-157{,}000 \times 64 &= -10{,}048{,}000 \\
&\ \overline{-25{,}578{,}400} = E\delta \text{ at } K \text{ and } I \\
-\ \ 38{,}000 \times 64 &= -\ \ 2{,}432{,}000 \\
&\ \overline{-28{,}010{,}400} = E\delta \text{ at } J
\end{aligned}
$$

FIG. 13 : 12

both the individual loads from Table 13 : 4 and the net load at each joint. The figure shows also the reactions on the phantom beam and the shear on each section between joints. The computations of the bending moments are also shown in the figure. These bending moments are numerically equal to the deflections of the corresponding joints on the lower chord of the truss multiplied by 0.001 E. In the case of joint H the agreement between this value as computed by elastic weights and by the method of work is exact.

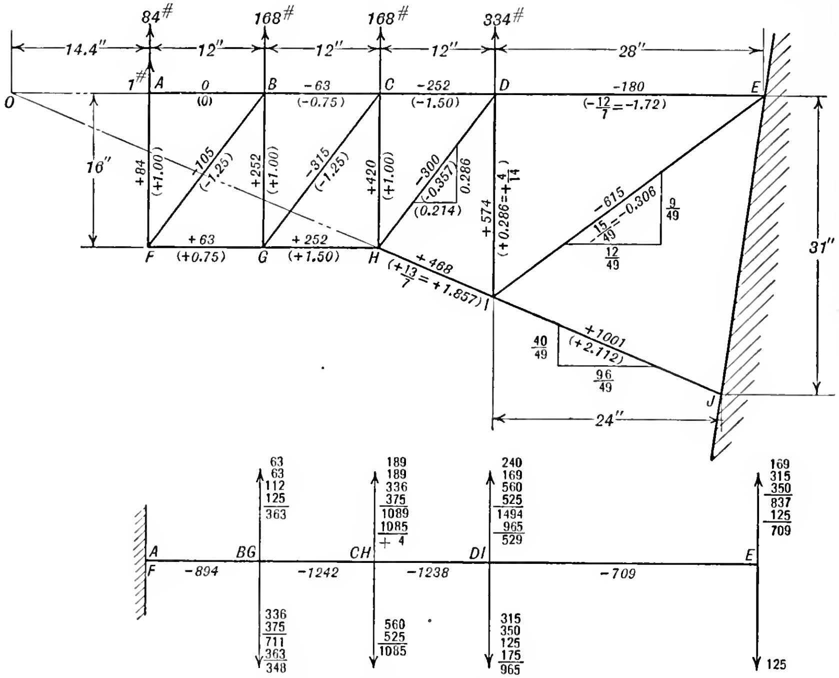
Considerable space and some time could have been saved in the elastic weight computations of this example if advantage had been taken of the symmetry of the truss to work on only one half of it. This was not done as it was considered desirable to work the problem out completely. Moreover, the symmetry would not have helped in the method of work computations.

TABLE 13 : 4

SIMPLY SUPPORTED TRUSS — COMPUTATION OF ELASTIC WEIGHTS

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Member	0.001 P	L	A	$\frac{E\Delta L}{1000}$	M.C.	R	S.M.C.	h	Sin θ	r	W_c	W_w	Sign	P.A.	S_H	S_HL
BC	−73.8	100	3.00	2460.0	L	48	..	..	..	..	51.25	..	+	L	+0.270	−664.200
CE	−125.0	128	6.67	2400.0	J	48	..	..	..	..	50.00	..	+	J	+0.750	−1800.000
EF	−73.8	100	3.00	2460.0	H	48	..	..	..	..	51.25	..	+	H	+1.230	−3025.800
AL	+36.9	72	1.5	1771.2	B	48	..	..	..	..	36.90	..	+	B	−0.135	−239.112
LJ	+112.2	128	3.74	3840.0	C	48	..	..	..	..	80.00	..	+	C	−0.510	−1958.400
JH	+112.2	128	3.74	3840.0	E	48	..	..	..	..	80.00	..	+	E	−0.990	−3801.600
HG	+36.9	72	1.50	1771.2	F	48	..	..	..	..	36.90	..	+	F	−0.615	−1089.288
AB	−61.5	60	3.13	1180.8	..	..	L	72	0.8	57.6	..	20.50	↶	AL	+0.225	−265.680
BL	+61.5	60	3.13	1180.8	..	..	A	72	0.8	57.6	..	20.50	↶	AL	−0.225	−265.680
LC	−48.0	80	2.50	1536.0	..	..	K	64	0.6	38.4	..	40.00	↶	LK	+0.300	−460.800
CK	+19.2	48	1.20	768.0	..	E. W. = +24.00 at K; −12.00 at L and J									0	0
CJ	+16.0	80	1.33	960.0	..	..	K	64	0.6	38.4	..	25.00	↶	KJ	−0.300	−288.000
DJ	−0	48	1.00	0	..	..	..	..	..	..	..	..	..		0	0
JE	+16.0	80	1.33	960.0	..	..	I	64	0.6	38.4	..	25.00	↷	JI	+0.300	+288.000
EI	+19.2	48	1.20	768.0	..	E. W. = +24.00 at I; −12.00 at J and H									0	0
EH	−48.0	80	2.50	1536.0	..	..	I	64	0.6	38.4	..	40.00	↷	IH	−0.300	+460.800
HF	+61.5	60	3.13	1180.8	..	..	G	72	0.8	57.6	..	20.50	↷	HG	−1.025	−1210.320
FG	−61.5	60	3.13	1180.8	..	..	H	72	0.8	57.6	..	20.50	↷	HG	+1.025	−1210.320
																−15,530.400

The second example is the computation of the deflections of the upper chord joints of the cantilever truss shown in Fig. 13 : 13 by elastic weights and the deflection of joint A by the Method of Work. The truss dimensions, external loads, and the axial loads in the truss members due to the external loads and to a unit load at joint A are shown on the figure. The sectional areas of the members are given



$$
\begin{array}{rcl}
-\ \ 709 \times -28 &=& 19{,}852 = E\delta \text{ at } D \\
-1238 \times -12 &=& 14{,}856 \\
 && \overline{34{,}708} = E\delta \text{ at } C \\
-1242 \times -12 &=& 14{,}904 \\
 && \overline{49{,}612} = E\delta \text{ at } B \\
-\ \ 894 \times -12 &=& 10{,}728 \\
 && \overline{60{,}340} = E\delta \text{ at } A
\end{array}
$$

FIG. 13 : 13

in Table 13 : 5, in which the computations of the elastic loads and the deflection of joint A by the Method of Work are tabulated. The lower part of Fig. 13 : 13 is a diagram of the phantom beam with the elastic loads on it shown, and the computations of the bending moments on the phantom beam. The results given are the values of $E\delta$ and would have to be multiplied by the value of E of the material used to obtain the true deflections. It may be noted that in this case also the agree-

ment between the two methods for computing the deflection of joint A is exact.

It may be noticed that Table 13 : 5 differs in some details from Table 13 : 4. An additional column, number 7, headed "H" is included. This is the horizontal distance from a member to its moment center. Also the column headed "Sin θ" is placed immediately after the "H" column instead of after "h." This is due to the fact that whereas in the first example all of the chord members were horizontal and their moment arms could be read directly from the truss diagram, in this second example two of the chord members are sloping and their moment arms must be computed from the formula $R = H \sin \theta$. Attention is also called to the fact that in the cases of the web members between non-parallel chords the parts of their elastic weights computed by the chord member rule and those computed by the web member rule are not combined until all the elastic weight forces acting at each joint on the phantom beam are added up.

As the truss is a cantilever fixed at one end, the phantom beam is treated as a cantilever supported at the free end of the truss. This is indicated by the diagram of the phantom beam. This being the case, the loads on the phantom beam at A and F were not shown in the diagram as they would have no effect on the bending moments on that beam.

The third and last illustrative example is that of a truss supported at two points, but with an overhanging cantilever. In this case the deflections of two points are checked by the Method of Work, one at the tip of the cantilever and the other between the two supports. The deflections computed are those of the lower chord joints.

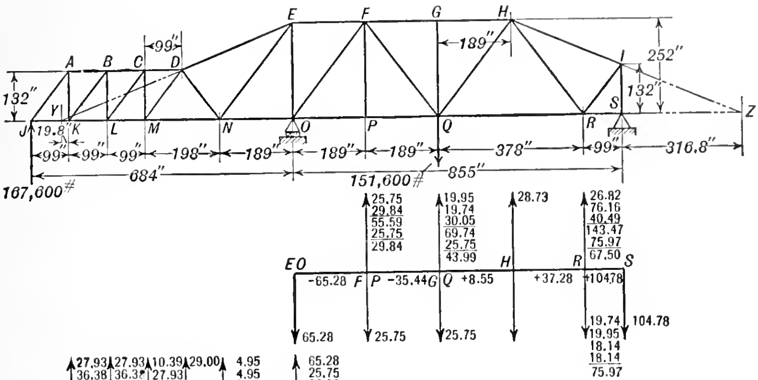
Most of the computations are carried out in the same manner as in the two preceding examples, but there are a few points that require special notice. The cantilever not being fixed at its supported end, but only restrained by the structure between the supports, this must be taken into account in the computations of the deflections by applying a load at the free end of the phantom beam for the cantilever portion of the truss equal to the reaction at that point on the simply supported span but in the opposite direction. In this case the force in question is an up load of 65.28 at EO. This step involves the question of whether the elastic weights listed as being applied at E and O should be considered as loads on the inner span or as loads on the cantilever. In practice it makes no difference which is done. In the example they are all shown in Fig. 13 : 14 as loads on the phantom beam of the cantilever. If the load of 14.83 representing the elastic weight of member EF had been placed at EO on the phantom beam of the inner

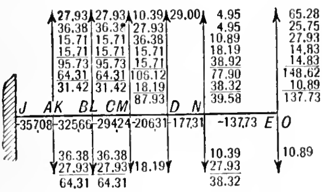
TABLE 13 : 5

COMPUTATION OF ELASTIC WEIGHTS — CANTILEVER TRUSS

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Member	P 1	L	A	$E\Delta L$	M.C.	H	Sin θ	R	S.M.C.	h	r	W_c	W_w	Sign	P.A.	S_A	$s_A E\Delta L$
AB	0	12	3/4	0	F	..	..	16	..	..	..	0	..	..	..	0	0
BC	−63	12	3/4	−1008	G	..	..	16	..	..	..	63	..	+	G	−3/4	+756
CD	−252	12	1	−3024	H	..	..	16	..	..	..	189	..	+	H	−3/2	+4536
DE	−180	28	1	−5040	I	..	..	21	..	..	..	240	..	+	I	−12/7	+8640
FG	+63	12	3/4	+1008	B	..	..	16	..	..	..	63	..	+	B	+3/4	+756
GH	+252	12	1	+3024	C	..	..	16	..	..	..	189	..	+	C	+3/2	+4536
HI	+468	13	13/7	+3276	D	50.4	5/13	252/13	..	..	..	169	..	+	D	+13/7	+6084
IJ	+1001	26	143/28	+5096	E	78.4	5/13	392/13	..	..	..	169	..	+	E	+104/49	+10816
AF	+84	16	1	+1344	..	..	1	..	B	12	12	..	112	↶	AB	+1	+1344
FB	−105	20	7/4	−1200	..	..	4/5	..	A	12	9.6	..	125	↶	AB	−5/4	+1500
BG	+252	16	1	+4032	..	..	1	..	C	12	12	..	336	↶	BC	+1	+4032
GC	−315	20	7/4	−3600	..	..	4/5	..	B	12	9.6	..	375	↶	BC	−5/4	+4500
CH	+420	16	1	+6720	..	..	1	..	D	12	12	..	560	↶	CD	+1	+6720
HD	−300	20	25/21	−5040	0	50.4	4/5	40.32	C	12	9.6	125	525	↶	CD	−5/14	+1800
DI	+574	21	41/30	+8820	0	50.4	1	50.4	E	28	28	175	315	↶	DE	+4/14	+2520
IE	−615	35	205/56	−5880	0	78.4	3/5	47.04	D	28	16.8	125	350	↶	DE	−15/49	+1800
																$E\delta_A = 60340$	

span, the reaction at that point would have been 14.83 larger than that shown, and when this increased reaction, reversed, was put on the loading diagram for the phantom beam of the cantilever the net load at EO on that beam would be the same as shown. The same would be true of any of the other elastic weights listed as applied at E or O.





$$
\begin{aligned}
-65.28 \times 189 &= -12{,}338 = E\delta \text{ at } P\\
-35.44 \times 189 &= -\underline{6{,}698}\\
& -19{,}036 = E\delta \text{ at } Q\\
+8.55 \times 189 &= +1{,}616\\
+37.28 \times 189 &= +\underline{7{,}046}\\
& -10{,}374 = E\delta \text{ at } R\\
+104.78 \times 99 &= +10{,}373\\
\text{Error} &= -\phantom{10{,}37}1 \text{ in } E\delta \text{ at } S
\end{aligned}
$$

$$
\begin{aligned}
-137.73 \times -189 &= +\ 26{,}031 = E\delta \text{ at } N\\
-177.31 \times -99 &= +\ 17{,}554\\
-206.31 \times -99 &= +\ \underline{20{,}425}\\
& +\ 64{,}010 = E\delta \text{ at } M\\
-294.24 \times -99 &= +\ \underline{29{,}130}\\
& +\ 93{,}140 = E\delta \text{ at } L\\
-325.66 \times -99 &= +\ \underline{32{,}240}\\
& +125{,}380 = E\delta \text{ at } K\\
-357.08 \times -99 &= +\ \underline{35{,}351}\\
& +160{,}731 = E\delta \text{ at } J
\end{aligned}
$$

FIG. 13 : 14

The elastic weights of members DN and NE show the necessity of inserting the phrase "as indicated by the character of the load in the member" in the statement of the rule for the direction of the moment part of the elastic weight for a web member. The shear on the index sections of these members is positive, but the compression in DN and tension in NE indicate negative shear, the vertical component of the load in the chord member DE being greater than the shear on these sections. The moment portion of the elastic weights of these two members is therefore clockwise instead of counter-clockwise.

In this example the check between the methods of elastic weights

TABLE 13 : 6

Computation of Elastic Weights — Simple Truss With Overhang

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1
Member	$\frac{P}{1000}$	L	A	$E\Delta L$	M.C.	H	Sin θ	R	S.M.C.	h	r	W_c	W_w	Sign	P.A.	S_j	$S_jE\Delta L$	S_q	$S_qE\Delta L$	Member
AB	−125.7	99	6	2074	*K*	..	..	132	..	..	..	15.71	..	+	*K*	−0.750	+1,555	0	0	*AB*
BC	−251.4	99	12	2074	*L*	..	..	132	..	..	..	15.71	..	+	*L*	−1.500	+3,111	0	0	*BC*
CD	−377.0	99	18	2074	*M*	..	..	132	..	..	..	15.71	..	+	*M*	−2.250	+4,666	0	0	*CD*
DE	−518.7	312	26	6224	*N*	415.8	5/13	159.9	..	..	..	38.92	..	+	*N*	−3.095	+19,263	0	0	*DE*
EF	−454.9	189	23	3738	*O*	..	..	252	..	..	..	14.83	..	+	*O*	−2.714	+10,145	0	0	*EF*
FH	−380.6	378	19	7572	*Q*	..	..	252	..	..	..	30.05	..	+	*Q*	−1.514	+11,464	+0.8368	−6,336	*FH*
HI	−124.5	312	6	6474	*R*	415.8	5/13	159.9	..	..	..	40.49	..	+	*R*	−0.495	+3,205	+0.2737	−1,772	*HI*
JK	+125.7	99	6	2074	*A*	..	..	132	..	..	..	15.71	..	+	*A*	+0.750	+1,556	0	0	*JK*
KL	+251.4	99	12	2074	*B*	..	..	132	..	..	..	15.71	..	+	*B*	+1.500	+3,111	0	0	*KL*
LM	+377.1	99	18	2074	*C*	..	..	132	..	..	..	15.71	..	+	*C*	+2.250	+4,667	0	0	*LM*
MN	+502.7	198	26	3828	*D*	..	..	132	..	..	..	29.00	..	+	*D*	+3.000	+11,484	0	0	*MN*
NO	+454.9	189	23	3738	*E*	..	..	252	..	..	..	14.83	..	+	*E*	+2.714	+10,145	0	0	*NO*
OQ	+417.8	378	21	7520	*F*	..	..	252	..	..	..	29.84	..	+	*F*	+2.114	+15,897	−0.4184	−3,146	*OQ*
QR	+229.8	378	12	7239	*H*	..	..	252	..	..	..	28.73	..	+	*H*	+0.914	+6,616	−0.5053	−3,658	*QR*
RS	0	99	10	0	*I*	..	..	132	..	..	..	0	..	+	*I*	0	0	0	0	*RS*
JA	−209.5	165	12	2881	..	..	4/5	..	*K*	99	79.2	..	36.38	↶	*JK*	−1.250	+3,601	0	0	*JA*
AK	+167.6	132	8	2765	..	..	1	..	*J*	99	99	..	27.93	↶	*JK*	+1.000	+2,765	0	0	*AK*
KB	−209.5	165	12	2881	..	..	4/5	..	*L*	99	79.2	..	36.38	↶	*KL*	−1.250	+3,601	0	0	*KB*
BL	+167.6	132	8	2765	..	..	1	..	*K*	99	99	..	27.93	↶	*KL*	+1.000	+2,765	0	0	*BL*
LC	−209.5	165	12	2881	..	..	4/5	..	*M*	99	79.2	..	36.38	↶	*LM*	−1.250	+3,601	0	0	*LC*
CM	+167.6	132	8	2765	..	..	1	..	*L*	99	99	..	27.93	↶	*LM*	+1.000	+2,765	0	0	*CM*
MD	−209.5	165	12	2881	..	..	4/5	..	*N*	198	158.4	..	18.19	↶	*MN*	−1.250	+3,601	0	0	*MD*
DN	−39.9	165	4	1646	*Y*	415.8	4/5	332.6	*M*	198	158.4	4.95	10.39	↷	*MN*	−0.238	+392	0	0	*DN*
NE	+39.9	315	4	1646	*Y*	415.8	4/5	332.6	*O*	189	151.2	4.95	10.89	↷	*NO*	+0.238	+392	0	0	*NE*
EO	+167.6	252	8	5279	..	..	1	..	*N*	189	189	..	27.93	↶	*NO*	+1.000	+5,279	0	0	*EO*
OF	+61.8	315	5	3893	..	..	4/5	..	*P*	189	151.2	..	25.75	↷	*OP*	+1.000	+3,893	+0.6974	+2,715	*OF*
FQ	−61.8	315	5	3893	..	..	4/5	..	*P*	189	151.2	..	25.75	↷	*PQ*	−1.000	+3,893	−0.6974	+2,715	*FQ*
QH	+265.3	315	14	5969	..	..	4/5	..	*R*	378	302.4	..	19.74	↷	*QR*	+1.000	+5,969	−0.5525	−3,298	*QH*
HR	−191.5	315	10	6032	*Z*	415.8	4/5	332.6	*Q*	378	302.4	18.14	19.95	↷	*QR*	−0.762	+4,596	+0.4211	−2,540	*HR*
IR	+191.5	165	10	6032	*Z*	415.8	4/5	332.6	*S*	99	79.2	18.14	76.16	↷	*RS*	+0.762	+4,596	−0.4211	−2,540	*IR*
IS	−201.1	132	10	2655	*R*	99	1	99	..	..	..	26.82	..	+	*R*	−0.800	+2,124	+0.4421	−1,174	*IS*
																	+160,718		−19,034	

and work is not exact though it is very close. The difference is negligible and is due to the fact that only a limited number of significant figures were used in the computations.

The methods of tabulating the computations illustrated in the above numerical examples is recommended for general use. In special cases it will be possible to omit some of the columns as they would be blanks or all the quantities listed would be the same. For most cases, however, it is advisable to include them all so that all of the computations will be shown in the tables.

If a check by the method of work is made at any one point and the deflection obtained is not the same as that obtained by elastic weights, the elastic weight of each member should be checked. The best way to do this is to consider that member by itself and find the moment on the phantom beam of that one weight and its reactions. If the value obtained checks the value of $s\Delta L$ in the method of work computations, that elastic weight is correct. If it does not check, either the elastic weight of the member or the value of s used in the method of work computations is in error. The student should check several of the elastic weights in the numerical examples to see how this is done.

13 : 12. Method of Elastic Weights for Beams — The general method just described for trusses can also be used for the determination of beam deflections, though the details of application differ. In the beam, the elements into which the structure is divided, instead of being bars of finite length subjected to axial load only, would be taken as slices of infinitesimal thickness, dx, normal to the axis of the beam. In the truss the deflection due to each element was $s\Delta L$ or sPL/AE. In the beam it would be $mMdx/EI$. The elastic weight corresponding to a single element would therefore be equal to Mdx/EI multiplied by the load whose bending moment curve on a phantom beam would be numerically equal to the influence line for m, the bending moment at the element.

For a beam simply supported at its ends, the influence line for bending moment at any point is numerically equal to the curve of bending moments on a phantom beam with the same supports and subjected to a load of unity at the point in question. As Mdx/EI for any element is the area of the portion of the M/EI curve over that element, the curve of elastic weights for the span will be numerically equal to the M/EI curve and the phantom beam will be considered as supported at the same points as the actual beam. On a cantilever fixed at one end the influence line for m will be numerically equal to the bending moment curve for a load of unity on a cantilever phantom beam supported at the free end of the actual beam. Hence for a cantilever

fixed at one end the M/EI curve will still represent the curve of elastic load but the phantom beam must be assumed to be supported at the free end of the actual cantilever. It may be noticed that the locations of the supports for the phantom beam are the same when the actual structure is a beam as for when it is a truss. An overhanging end of a beam with two supports would be treated in the same manner as the similar case of a truss except that the elastic weights of the cantilever portion of the beam would be the curve of M/EI instead of a system of concentrated loads each representing a truss member of finite length.

The practical application of this method to beams is so much like that of the method of Moment Areas that the two are sometimes considered identical. As the Moment Area method cannot be applied to trusses while the underlying theory of the method of Elastic Weights does apply to both trusses and beams, the two methods should be recognized as fundamentally different.

In the case of a cantilever beam with a fixed end, the application of the two methods is identical, as can be seen from the illustrative example of Arts. 13 : 8 and 13 : 9. For a beam supported at the ends the methods would differ very little. If the computations were made graphically, the portion b–n of the line a–b–n of Fig. 13 : 7 could have been omitted, and the straight line a–m would have been drawn as the closing line of the string polygon instead of as the line on a scale diagram of the elastic curve representing the desired reference line. This difference, however, is purely one in the mind of the computer and obviously makes no difference in the practical application of the method.

It might be noted that if the ray P–Q were drawn in the force polygon of Fig. 13 : 6, it would give the values of 9–Q and Q–0 the reactions on the phantom beam, but as the string polygon from which the deflections are to be measured has been drawn by the time this is possible, the numerical values of these reactions are of no use unless they must be used as loads on the phantom beams of overhanging ends of the actual beam.

The analytical computations by the method of elastic weights for a beam simply supported at the ends and with an M/EI curve of irregular shape would be identical with those by the method of Moment Areas, except that the figures in column 11 of Table 13 : 3 would be interpreted as the part of the total moment at each station due to the left reaction on the phantom beam. Column 12 would then be the net bending moment on the phantom beam due to the combined action of that reaction and the distributed elastic load represented by the M/EI curve. In cases where the M/EI curve is of such a regular shape

that the reactions on the phantom beam can be easily computed, this would be done first and the moments on the phantom beam, and thus the deflections, then worked out in a single process.

13 : 13. Truss Deflections by the Williot Diagram — A much used graphical method for computing the deflections of a truss is that of the Williot Diagram, named after the French engineer who originated it. While the methods of deflection computation considered above are all algebraic, both in derivation and application, this method is entirely geometrical.

Let AC and BC in Fig. 13 : 15 be any two members of a truss, and let the changes in length of both of these members and the deflections of A and B be known. The problem is to determine the deflection of C. Let A–A' be the deflection of A, and B–B' the deflection of B, C–$C1$ the change in length of AC, assumed to be a shortening, and C–$C2$ the change in length of BC, assumed to be a lengthening. The true deflected position of C will be $C7$ the intersection of two arcs, one with its center at A' and a radius equal to A–$C1$, and the other with its center at B' and its radius equal to B–$C2$. The movement of point C from its original position to $C7$ may be considered as the resultant of either of two sets of three steps depending upon whether we consider it the outer end of AC or of BC. Let us consider it first as the outer end of AC. As point A moves to A', member AC may be considered to move parallel to itself, bringing C to $C3$, C–$C3$ being equal and parallel to A–A'. The change in length of AC then brings it to $C5$, $C3$–$C5$ being equal to C–$C1$. Finally it moves along the arc $C5$–$C7$ of radius A–$C1$ = $A' - C5$ to the point $C7$ the only point on that arc consistent with its movement as the outer end of BC. Similarly, considering C as the outer end of BC, it may be considered to move to $C4$ due to the movement of point B, then to $C6$ due to the change in length of BC, and finally along the arc $C6$–$C7$ to its intersection with $C5$–$C7$.

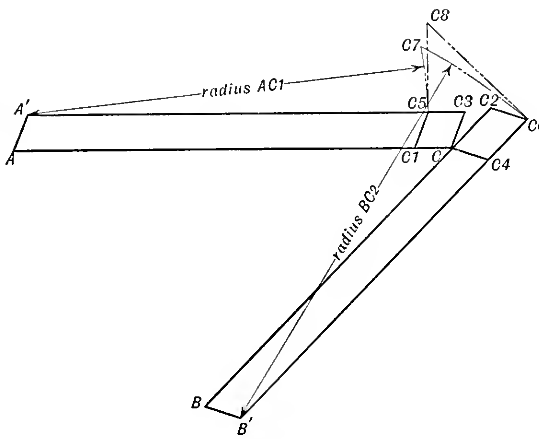


Fig. 13 : 15

Theoretically this method could be extended to determine the deflec-

tions of all of the joints of a truss, but in practice the changes in length and deflections are so small in comparison with the actual lengths of the members that it would be impractical. If, however, instead of drawing the arcs $C5$–$C7$ and $C6$–$C7$, we draw $C5$–$C8$ and $C6$–$C8$ perpendicular to AC and BC respectively we would get an intersection $C8$ that would be practically coincident with $C7$ where C–$C1$ and C–$C2$ are as small in comparison with AC and BC as they are in practical structures. The advantage of doing this is that when the perpendiculars representing the chords are used instead of the arcs, the deflection of C can be determined without any knowledge of the lengths of AC and BC but only that of their directions and changes in length and the deflections of A and B. This is demonstrated in Fig. 13 : 16 as follows:

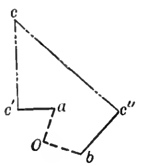


FIG. 13 : 16

From any point 0 lay off Oa equal and parallel to A–A' (or C–$C3$) the deflection of A, and Ob equal and parallel to B–B' (or C–$C4$) the deflection of B. Points a and b then represent the deflected positions of A and B, with O as the origin. Now lay off a–c' parallel to AC and equal to its change in length C–$C1$ (or $C3$–$C5$) and b–c'' parallel to BC and equal to its change in length C–$C2$ (or $C4$–$C6$). Both of these distances should be laid off in the direction of the motion of the point C when considered as a point on the bar in question. From c' and c'' draw c'–c and c''–c perpendicular to AC and BC respectively. The intersection c of these two lines will represent the deflected position of C and Oc will be the deflection of the point.

Figure 13 : 16 shows what might be termed the single unit of a Williot diagram, the diagram for a complete truss being made up of as many

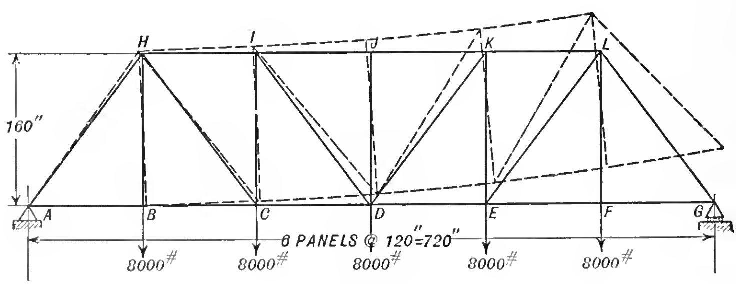


FIG. 13 : 17

such units as there are pairs of members of the truss, combined in a single diagram. The method of construction of the diagram for a complete truss can be demonstrated most conveniently by means of an example, and the computation of the deflection of the truss shown in Fig. 13 : 17 will be used for this purpose. The first step is to com-

pute the loads in the truss members and the corresponding values of ΔL or $E\Delta L$. These computations are tabulated in Table 13 : 7. The changes in length are expressed in terms of ΔL multiplied by 0.001 E, the factor 0.001 being used to get the decimal point in a convenient location. Any similar factor could be used, the proper correction being made in the scale of the drawing. In order to start the construction, it is necessary to select a point, the deflection of which can be assumed equal to zero, and a member radiating from that point, the direction of which will be assumed unchanged by the distortion of the truss. In this case as the left hand support point A is fixed to the abutment we will assume it to be the point of zero deflection, and will choose the member AB as the one assumed to be unchanged in slope, though we might have chosen AH.

TABLE 13 : 7

Member	P	L	A	$E\Delta L$
AB	+15,000	120	0.6302	+2860
BC	+15,000	120	0.6302	+2860
CD	+24,000	120	0.6954	+4145
DE	+24,000	120	0.6954	+4145
EF	+15,000	120	0.6302	+2860
FG	+15,000	120	0.6302	+2860
HI	−24,000	120	1.1800	−2440
IJ	−27,000	120	1.1800	−2745
JK	−27,000	120	1.1800	−2745
KL	−24,000	120	1.1800	−2440
AH	−25,000	200	2.0985	−2385
BH	+8,000	160	0.6302	+2030
CH	+15,000	200	0.6302	+4760
CI	− 4,000	160	0.6302	−1015
DI	+5,000	200	0.6302	+1590
DJ	0	160	0.6302	0
DK	+5,000	200	0.6302	+1590
EK	−4,000	160	0.6302	−1015
EL	+15,000	200	0.6302	+4760
FL	+8,000	160	0.6302	+2030
GL	−25,000	200	2.0985	−2385

Upper Chord	3 1/4 × 0.120	A = 1.1800 sq. in.
Lower Chord C–E	2 3/4 × 0.083	A = 0.6954 sq. in.
Lower Chord AC and EG	2 1/2 × 0.083	A = 0.6302 sq. in.
End Posts	3 3/4 × 3/16	A = 2.0985 sq. in.
Web Verticals	2 1/2 × 0.083	A = 0.6302 sq. in.
Web Diagonals	2 1/2 × 0.083	A = 0.6302 sq. in.

Having computed the values of ΔL and selected A as the point of zero deflection and AB as the member of no change in slope, designate some point on the paper to represent both the original and final locations of point A, and label it a. From a lay off a length of 2860 units to

the right to a convenient scale. This distance is laid off to the right as AB is a tension member, and under load point B moves to the right with respect to A.

The point in question can therefore be labelled b. Knowing the deflections of points A and B, we can now use the construction of Fig. 13 : 16 to find the deflection of point H. From a lay off a distance of 2385 units parallel to AH and from b a distance of 2030 units parallel to BH in the directions shown in Fig. 13 : 18. These distances are laid off in the direction of the movement of the end H of each bar with respect to the end A or B the deflection of which has already been determined. At the ends of these distances construct perpendiculars to AH and BH respectively and find their intersection h. The distance from a (the origin) to h will then represent the deflection of point H. This construction is repeated to find the deflected positions of points C, I, D, J, K, E, L, F, and G in succession. In Fig. 13 : 18 the points designated by the lower case letters represent the deflected positions of the joints designated in Fig. 13 : 17 by the corresponding capital letters. In every case the deflection is to be measured from a.

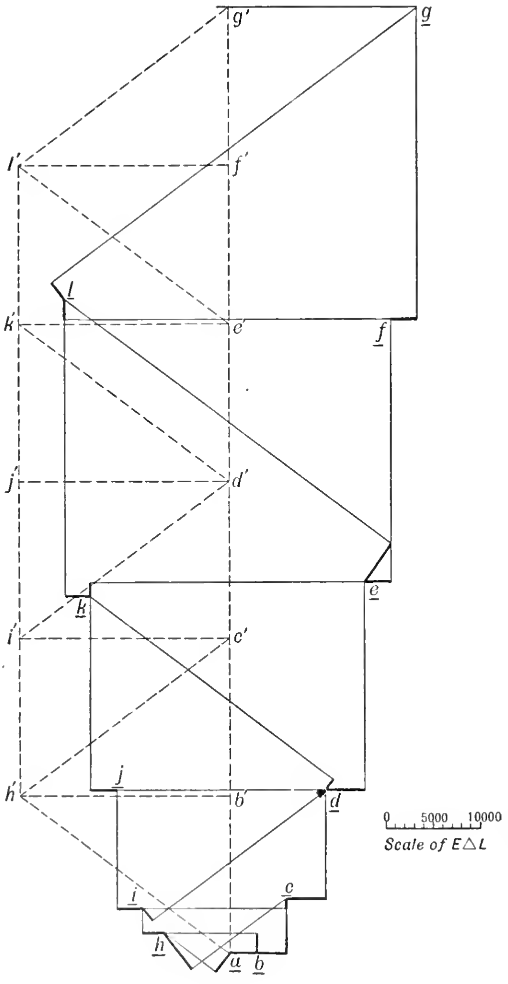


FIG. 13 : 18

It should be noticed that according to Fig. 13 : 18, the vertical component of the deflection of the right support, point G, is a considerable amount. This is due to the fact that AB which was assumed

fixed in direction actually rotates, and a correction must therefore be made if we wish to determine, not the deflections from the deflected locations of AB, but those from the line joining the two supports. The situation is very similar to that encountered in the use of the methods of Moment Areas on a simply supported beam where the first step of the computation gave the deflection from a tangent to the elastic curve of the beam when the deflection desired was that from the line joining the supports.

If the deflections from the uncorrected Williot diagram were used to plot the deformed position of the truss, it would be similar to that shown by the dotted lines in Fig. 13 : 17 in which the deflections are greatly exaggerated.

In order to get the deflections desired it is necessary to determine the deflections of the various joints assuming that the truss is rotated so that the right support G moves vertically a distance equal to the vertical component of the distance ag of Fig. 13 : 18. The desired deflection is then the geometric difference between the two deflections figured. In making this correction it is not necessary to consider the horizontal deflection of G, due to the rotation of the truss as it will be negligible in practical cases.

To determine the deflections of the truss joints assuming the truss to rotate without deformation until point G has deflected vertically a distance equal to the vertical component of ag, draw ag vertically through a, locating g' by the intersection of this line and a horizontal through g. Using ag' to represent the lower chord AG draw a scale diagram of the original truss, each member being represented by a line perpendicular to its direction in the original truss. The scale of this diagram, which is called the correction diagram, is determined by the length of ag'. The correction diagram for the truss of Fig. 13 : 17 is shown in Fig. 13 : 18 the various joints being designated by the same letters as in Fig. 13 : 17, but in lower case and with primes. Thus j' corresponds to J, etc. The actual deflection of any joint is then shown graphically by the distance between the point on the correction diagram and the point on the Williot diagram which refers to the joint in question.

It remains to prove that the correction diagram shows the deflections of the truss joints when the truss is rotated without deformation. As the truss rotates about A the movement of each joint will be equal to its distance from A multiplied by the angle of rotation in radians. The angle of rotation being very small in all practical cases we may assume it to be equal to its sine, $\frac{ag'}{AG}$. Since the lines representing any one truss member in the two figures are at right angles, the correc-

tion diagram of Fig. 13 : 18 and the truss diagram of Fig. 13 : 17 are geometrically similar. Therefore

$$\frac{ad'}{AD} = \frac{ag'}{AG} \text{ whence } ad' = AD \cdot \frac{ag'}{AG}.$$

Similarly
$$\frac{ak'}{AK} = \frac{ag'}{AG} \text{ whence } ak' = AK \cdot \frac{ag'}{AG}$$

and so on for the other joints of the truss.

In any practical case the angle of rotation will be so small that the angle and its sine may both be assumed equal to ag'/AG. Then as AD, AK, etc. are the true distances from A to the truss joints, ad', ak', etc. will be the movements of the various joints due to the rotation of the truss. This establishes the validity of the construction of the correction diagram. This diagram was originally devised by Professor Mohr and is generally known by his name.

In order to draw a Williot diagram it is necessary to assume one point fixed in location and one member fixed in direction. It is a simple matter to find a point the location of which may be assumed fixed, but it is seldom that there will be a member of which it can be determined beforehand that its direction will be unchanged by the deformation of the truss. When that does occur, however, as in the case of a cantilever truss with a member at the inner end fixed in direction, the Williot diagram will give the true deflection without being supplemented by a Mohr correction diagram. In the typical case of a truss supported at two points, one end will be supported against movement both horizontally and vertically, while the other point will be supported against vertical movement only. In the case of an airplane wing truss-type spar, for instance, the truss may be supported by a pin at the side of the fuselage or cabane, and an interplane strut near the wing tip. In this case the support at the fuselage or cabane would be considered fixed in both directions, while the other support would be assumed to be held against vertical motion but free to deflect horizontally. The Williot diagram would therefore be drawn starting with the inner support as the fixed point.

If, in the numerical example given above, the member AH had been assumed fixed in direction instead of AB, the Williot diagram would have been considerably different from that shown, but the Mohr correction diagram would have resulted in the net deflections indicated being the same. If G had been chosen as the fixed point, the vertical deflections would have been the same as those shown, but the horizontal deflections would all have differed from those shown by the distance $g'g$

the horizontal component of the deflection of G or the movement of G relative to A.

13 : 14. Stresses in Trusses Subjected to Combined Lateral and Axial Loads — The airplane designer is often called upon to design long shallow trusses subjected to axial as well as lateral loads. The typical case of this class is a metal wing spar truss in an externally braced cellule. When a beam with a continuous web is used for a wing spar, the stresses due to combined bending and compression can be computed directly by the use of the formulas of Chapter XI. These formulas cannot, however, be applied directly to shallow trusses subjected to combined bending and compression as the truss has no true moment of inertia and therefore the fundamental quantity L/j cannot be determined.

It is sometimes suggested that a satisfactory and conservative "effective" value of I can be obtained by neglecting the web members of the truss and adding the areas of the cross-sections of the chords each multiplied by the square of the distance from its centroid to the centroid of the pair considered as a unit. The great objection to this procedure is that it is far from conservative since if this "moment of inertia of the chords alone" is used in the formulas for deflection, the resulting quantities will represent only that part of the total deflections which is due to the length changes of the chord members. Consequently both the deflections and the total bending moments computed by this method will be too small.

The best procedure for determining the stresses in a truss subjected to combined bending and compression is to build a section of the truss and measure its deflections at several points under a lateral load large enough to produce measurable deflections, but not large enough to stress any of the members beyond the elastic limit. The load and deflections being known, satisfactory "effective EI" values can be obtained by substituting in the formulas of Art. 13 : 1. The average of these values can then be used in the formulas of Art. 11 : 5. For preliminary design, a similar procedure may be followed, except that instead of determining the deflections under a known lateral load by test, these quantities would be computed by one of the methods described above. Since it is usually necessary to compute values of effective EI from deflections at several points along the span and use the average, the method of elastic weights will prove the simplest method of computation for most cases.

As the bending moments and shears at any section can be obtained from the formulas of Art. 11 : 5, the method recommended for computing the loads in the individual truss members involves the use of these

quantities. To obtain the loads in the chord members, divide the axial compression between the chords in proportion to their sectional areas. To this quantity add or subtract, depending on the chord considered and the sign of the bending moment, the total bending moment on the truss divided by the distance between the centroids of the chords, (the "effective depth" of the truss). For parallel chord trusses the loads in the web members can be computed by multiplying the shear as found from the formulas of Art. 11 : 5 by the secant of the angle between the web member and the axis of the beam. For non-parallel chord trusses this quantity will have to be corrected to allow for the components normal to the spar axis of the stresses in the chord members.

In order to illustrate the proper method of computing the effective EI of a truss and to demonstrate the error involved in using the moment of inertia of the chords alone, the effective moment of inertia of the truss shown in Fig. 13 : 2 as indicated by the deflection of joint $L5$ will be determined. The deflection of this joint was found in Art. 13 : 3 to be 0.0668 inches when the truss was loaded with the equivalent of $100/36 = 2.78$ lb. per in. uniformly distributed. From Case 4 of Table 13 : 1 the deflection formula to use is

$$\delta = \frac{wx}{24\,EI}\,(L^3 - 2\,Lx^2 + x^3)$$

Rearranging, substituting the known values, and solving for EI, we have

$$EI = \frac{2.78 \times 144}{24 \times 0.0668}\,(216^3 - 2 \times 216 \times 144^2 + 144^3)$$

$$EI = 1{,}026{,}000{,}000$$

As E in this case was 29,000,000, the effective moment of inertia would be 35.4 in.^{4} Since both chords of this truss were $3/4 \times 0.035$ in. tubes with cross-sectional areas of 0.07862 sq. in., and the distance between them was 36 in., the moment of inertia of the chords alone would be $2 \times 0.07862 \times 18^2 = 51.0$ in.^{4}, a much larger value.

This computation shows how great an error may result from any attempt to compute an effective I from the properties of the chords alone, and also that the effect of the webs is to *increase* the deflection and *decrease* the effective EI. The explanation of this is simple. As stated in Art. 13 : 1, the formulas in Table 13 : 1 apply strictly only to the deflection due to bending, and the deflection due to shear must either be separately computed or approximated by using a reduced value of E. In a truss, the deflection corresponding to that in a beam due to shear, the deflection due to the tensions and compressions in the

web members, is normally a much greater part of the total deflection. The formulas of Table 13 : 1 used with the moment of inertia of the chords alone would give correct values for that part of the deflection due to the length changes of the chord members, but in the truss the deflection due to the length changes of the web members is too large to be neglected or approximated in the same manner as the shear deflection of a beam. Thus if we had used the deflection formula from Table 13 : 1 and $I = 51.0$, the value obtained for $E\delta$ would have been

$$E\delta = \frac{35.4}{51.0} \times 0.0668 \times 29{,}000{,}000 = 1{,}341{,}000$$

Adding the values from column 6 of Table 13 : 2 representing the effect of the changes in length of the chords alone, we obtain 1,373,937, which is in good agreement with the deflection computed from the beam formula and the moment of inertia of the chords alone. The remaining values of column 6 show that $E\delta$ due to the length changes of the web members amounts to 559,713, approximately 40 per cent of the $E\delta$ from the chords. The ratio of the "web deflection" to the "chord deflection" would have been much greater had smaller or more highly stressed web members been used, and it should be obvious that for this type of structure the "web deflection" must be properly computed if correct results are to be obtained.

Another objection to the use of the moment of inertia of the chords alone is that usually the size of each chord varies along its length, in which case additional assumptions have to be made in order to obtain a value of I for use.

In design, when the effective EI of a truss is needed for use in the formulas of Art. 11 : 5, it is essential that the effective I be computed for each span between supports, and desirable that the figure used for each span be the average of values obtained from the measured or computed deflections of at least three points in the span. The loading used for determining the effective EI need not be the same as that for which the truss must be finally designed, but each span may be assumed to be simply supported and subjected to two concentrated loads at joints at or between the third points of the span.

PROBLEMS

13 : 1. Determine the deflection of point J in the truss shown in Fig. 13 : 12 by the method of work. Use the loads shown on the figure and take the areas of members as given in Table 13 : 4.

13 : 2. Determine the deflection of all panel points on the lower chord of the truss shown in Fig. 13 : 12 due to a load of 10,000 lb. acting downward at J. From the data thus obtained draw an influence line for an upward reaction to be introduced at point J. What would the magnitude of this reaction be if the truss were loaded as shown in Fig. 13 : 12?

13 : 3. Determine the deflection of point b in Fig. 3 : 7 from the tangent to the elastic curve at f. Assume the EI of the beam to be 20,000,000 lb. in.^{2} and use the method you deem most simple.

13 : 4. Check the deflection of the truss shown in Fig. 13 : 14 by using the Williot Diagram.

CHAPTER XIV

STATICALLY INDETERMINATE STRUCTURES

As stated in Chapter III there are certain types of structure in which the outer forces, the reactions, or the inner forces, the loads in the various parts of the structure, cannot be determined from the conditions of equilibrium alone. Such structures are known as statically indeterminate or redundant structures since there are more unknown magnitudes, directions, and points of application of the reactions, or more internal forces on the elements of the structure than there are independent equations that can be written by the application of the conditions of statics. In this chapter some of the more important methods of analyzing structures of this type will be described and illustrated.

14 1. Degree of Redundancy — In the case of any structure, the relationships between the forces in the structure imposed by the conditions of static equilibrium can be represented by algebraic equations. Each condition of equilibrium when applied to the whole or any part of a structure can be represented by one and only one independent equation. If the structure is statically determinate the number of independent equations that can be obtained in this way will be equal to the number of unknown quantities appearing in them. All of the reactions on or stresses in the structure can then be found by solving the equations. If the equations deducible from the conditions of equilibrium are written for a redundant structure and its parts, it will be found that there will be n less independent equations than there are unknown quantities appearing in them. The structure is then said to be statically indeterminate or redundant in the nth degree.

It will sometimes happen that $p + r$ equations can be written for a structure on the basis of the conditions of equilibrium, and that a group of these equations p in number will contain only p unknowns. The remaining r equations will contain $r + n$ unknowns in addition to those appearing in the p equations. This indicates that the parts of the structure represented by the p equations are statically determinate and can be solved directly, while the remainder represented by the r equations are redundant in the nth degree. In such cases it is necessary for the complete analysis of the structure to obtain n new independent equations which will be based, not on the requirements for equilibrium, but on some other principle.

It is not always easy to determine the degree of redundancy of a structure and very often the designer will write down equations that are mutually dependent in the belief that they are independent, and be led to the belief that the structure is determinate when it is redundant, or that it is redundant to a lesser degree than it is in fact. When this is done certain of the equations will reduce to an identity or to equations which are almost identical, the latter being due to lack of precision of the arithmetical work such as caused by dropping small decimals. In such cases, one of the mutually dependent equations must be thrown out. The only sure way of determining the degree of redundancy is to write down the equations based on the conditions of equilibrium and solve them as far as possible, eliminating such equations as prove to be dependent. The number of unknowns that cannot be eliminated by the methods of algebra will then indicate the degree of redundancy.

14 : 2. Methods of Solving Indeterminate Structures — The simplest method of determining the values of the redundant unknowns is to make sufficient arbitrary assumptions. A good example of this is the common procedure of assuming a truss to be composed of members connected by frictionless pins, thus fixing the directions and points of application of the loads acting at the ends of the various members. Another is the assumption that a continuous beam acts as a series of disconnected simple beams, each resting on only two supports. This is a highly approximate method, which should be used only when great accuracy is not desired, or when it is known that the assumptions used for the structure being analyzed will give sufficiently precise results. Some assumptions of this character, however, are practically always needed, and most structures that are analyzed as statically determinate are really redundant structures in which the redundant unknowns have been eliminated by the use of simplifying assumptions.

The rational method of computing the redundant unknowns is to apply the principle of consistent deformations. This principle is: In any redundant structure, the distribution of stresses will be such that not only are the conditions of equilibrium satisfied, but also that the deformations of all parts of the structure will be consistent with respect to each other. The application of this principle may take any one of several forms, but no matter which method is used, if it is used correctly, the same stress distributions will be found in the various members. Its chief disadvantage is that, since the deformation of any member is a function of the area and stress in the member, the size of the members of the structure must be determined before the stresses in them can be computed. In certain special cases the principle of consistent deformations can be used for members of unknown size if it is

known that they are to be of constant section or that the values of EI and AE of the different members will be in previously determined ratios.

One application of the principle of consistent deformations was made in Chapter IV in the development of the Three-moment Equations from the proposition that the slope of the elastic curve just to the left of a support is necessarily the same as its slope just to the right of the same support.

14 : 3. The Included Statically Determinate Structure — Any statically indeterminate structure can be reduced to a statically determinate system by the removal of sufficient parts to make the number of unknowns equal to the number of independent equations derivable from the conditions of equilibrium. In the case of structures statically indeterminate with respect to the outer forces, this process would involve one or both of the following operations: complete removal of certain supports, and modification of the structure at certain supports so that the directions or points of application of the reactions would be fixed instead of unknown. The second of these operations includes any modification fixing a definite relationship between the components of the reactions affected or a modification that prevents moments being developed as part of the reactions. In general these operations either reduce the number of unknowns directly or provide additional "equations of condition" and thus reduce the degree of redundancy. Structures statically indeterminate with respect to the inner forces can similarly be made determinate by the omission or cutting of members, or the fixation, by the use of pinned connections, of the relations between the bending moments at the ends of certain members and the axial loads in them. Usually the pins are assumed to be placed on the axes of the members, so the end moments will be zero. The structure remaining after the changes that are necessary to make the number of unknowns equal to the independent equations of equilibrium have been made, may be called the "included statically determinate structure" or more briefly the "included determinate structure."

Suppose that, when removing parts from the redundant structure to form the included determinate one, we substitute for each part removed, external loads acting on the part remaining, these loads being identical with those previously exerted as internal forces by the part removed. The load distribution in, and the deformations of, the included determinate structure would then be identical with those of the redundant structure. The computer now has a statically determinate structure to deal with, but, the external loads representing the effects of the parts assumed removed being still unknown, he cannot analyze it by applying the conditions of equilibrium alone.

Normally there will be as many points on the redundant structure, and thus on the included determinate structure, the deflections of which are known, as there are redundant unknown external loads on the latter. If the deflections of these points of the included determinate structure can be expressed in terms of the redundant loads and equated to their known values, the equations needed for the evaluation of the redundant unknowns may be obtained.

Usually there are a number of alternative included statically determinate systems in any redundant structure, but the final results will be the same whichever system is used in the stress and deflection computations. The computer tries to select the one for which the numerical work will be the least tedious and, if possible, that on which the effect of the unknown external loads will be a minimum. Care must be taken, however, that the included structure selected for the computations is truly statically determinate, as it is easy to try to use one, part of which is unstable and part indeterminate.

14 : 4. The Principle of Super-Position — In general the deflections of the included determinate structure can be expressed as functions of sufficient simplicity to permit the direct computation of the unknown external loads on it, only when the structure is one to which the principle of super-position applies. This principle states that the net loads in and deflections of the structure under a system of external loads acting simultaneously are the algebraic sums of the stresses and deflections caused by each of the loads in the system acting alone. It applies to practically all airplane structures except those including members subjected to combined bending and axial loads. Even in this case, the error involved in assuming that the principle applies is small when the axial load is one of compression and small in comparison with the Euler load for the member. In practice, therefore, although the methods described in this chapter depend on the principle of super-position for their validity, they are frequently used in the analysis of structures including members subjected to combined axial and bending loads to which this principle does not apply directly.

When the principle of super-position applies, the axial load in any member of a structure may be expressed by the formula

$$P = P_0 + X_a P_a + X_b P_b + \cdot \cdot \cdot \qquad 14:1$$

where P = axial load in the member.

P_0 = axial load in the member due to the known external loads.

X_a = the unknown external load at some point a.

P_a = axial load in the member due to unit external load at a in the direction of X_a.

X_b = the unknown external load at some point b.
P_b = axial load in the member due to unit external load at b in the direction of X_b.

In the same manner the bending moment at any point may be expressed by

$$M = M_0 + X_aM_a + X_bM_b + \cdot \cdot \cdot \qquad 14:2$$

where M = the bending moment at the point.
M_0 = the bending moment due to the known external loads.
M_a = the bending moment due to unit external load at a in the direction of X_a, the unknown external load at that point.
M_b = the bending moment due to unit external load at b in the direction of X_b, the unknown external load at that point.

Similarly the deflection of any point may be expressed by

$$\delta = \delta_0 + X_a\delta_a + X_b\delta_b + \cdot \cdot \cdot \qquad 14:3$$

where δ = the deflection of the point.
δ_0 = the deflection due to the known external loads.
δ_a = the deflection due to a unit external load at a in the direction of X_a, the unknown external load at that point.
δ_b = the deflection due to a unit external load at b in the direction of X_b, the unknown external load at that point.

Formulas 14 : 1, 14 : 2 and 14 : 3 apply strictly only when all the unknown external loads are single forces, each acting at a point. If any of these external loads were couples acting on a section, the formulas could still be used for P_a, M_a, and δ_a considered as representing the effects of a unit couple at section a acting in the same sense as X_a, the unknown couple on that section.

14 : 5. Wilson's Method — The method developed by Prof. George Wilson[1] for computing the magnitudes of the reactions on structures that are statically indeterminate with respect to the external forces is particularly useful in problems involving continuous trusses and beams of varying section. If, as in the usual case, all the loads and reactions are co-planar and parallel, the included determinate structure is the beam or truss assumed to be resting on any two of its actual supports. The first step is to compute the values of δ_0 at the points of application of the other reactions. Then, assuming the structure supported at

[1] Proc. Roy. Soc., Vol. 62, Nov. 1897, and Morley "Strength of Materials," p. 218.

the same two points, the deflections of all the other points of support due to a unit load acting at each of them are computed, thus giving the desired values of δ_a, δ_b, . . . If, as in the usual case, the points of support are assumed to be fixed in position, the values of δ, the net deflections, will all be zero. If there are n such redundant supports, it is possible to write n equations of the form of 14 : 3 and solve for the unknown external forces, the redundant reactions. The true net reactions at the points of support of the included determinate structure can then be found by applying the equations of equilibrium.

If some of the supports are assumed to deflect under load by known or assumed amounts, such values would be used for δ in the equations of the form of 14 : 3, but otherwise the application of the method would be unaltered. If more than two equations of equilibrium were available for the determination of the reactions, as would be the case where the external forces were not co-planar, or were not parallel, the same general method would be used except that the included determinate structure would be chosen so that its reactions would have as many unknown properties as there were equations of equilibrium available for their determination.

The use of Wilson's Method for computing the reactions on a continuous beam is illustrated by the following computations for the beam shown in Fig. 14 : 1. For simplicity the moment of inertia and modulus of elasticity are assumed constant.

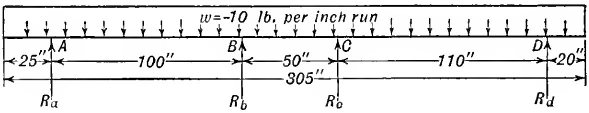


Fig. 14 : 1

The beam, assumed to be supported at A and D only, will be taken as the included determinate structure.

Taking moments about D

$$R_a \times 260 - 10 \times 305 \times 132.5 = 0$$

whence $R_a = 1555$ lb. and $R_d = 3050 - 1555 = 1495$ lb.

$$S_{-a} = -10 \times 25 = -250 \text{ lb.}$$
$$S_{+a} = -250 + 1555 = +1305 \text{ lb.}$$
$$M_a = -250 \times 12.5 = -3125 \text{ in.-lb.}$$

From Art. 13 : 1 (Chapter XIII),

$$EI\delta = x(x - L)\left[\frac{M_a}{2} + \frac{S_{+a}}{6}(x + L) + \frac{w}{24}(x^2 + xL + L^2)\right]$$

$$EI\delta_{bo} = 100 \times (-160)\left[-\frac{3125}{2} + \frac{1305}{6} \times 360 - \frac{10}{24}(100^2 + 100 \times 260 + 260^2)\right]$$

$$\delta_{bo} = 537{,}128{,}000/EI$$

$$EI\delta_{co} = 150 \times (-110)\left[-\frac{3125}{2} + \frac{1305}{6} \times 410 - \frac{10}{24}(150^2 + 150 \times 260 + 260^2)\right]$$

$$\delta_{co} = -558{,}032{,}000/EI$$

In these computations, δ_{bo} and δ_{co} are the deflections of B and C of the included determinate structure under the known external loads. In the same way δ_{bb} and δ_{cb} will be the deflections of B and C due to a unit load at B, while δ_{bc} and δ_{cc} will be the deflections due to a unit load at C.

Assume the included determinate structure to be subjected to a unit load acting upward at B. The deflections δ_{bb} and δ_{cb} can then be obtained from the formula given in Art. 13 : 1 (Chapter XIII),

$$\begin{aligned} EI\delta &= Wa(L - x)\,(2\,Lx - a^2 - x^2)/6\,L \\ EI\delta_{bb} &= 100 \times 160(2 \times 260 \times 100 - 100^2)/(6 \times 260) \\ \delta_{bb} &= 328{,}000/EI \\ EI\delta_{cb} &= 100 \times 110(2 \times 260 \times 150 - 100^2 - 150^2)/(6 \times 260) \\ \delta_{cb} &= 321{,}000/EI \end{aligned}$$

The deflections δ_{bc} and δ_{cc} can be computed from the effect of a unit load acting upward at C.

By the principle of reciprocal deflections,

$$\begin{aligned} \delta_{bc} &= \delta_{cb} = 321{,}000/EI \\ EI\delta_{cc} &= 150 \times 110(2 \times 260 \times 150 - 2 \times 150^2)/(6 \times 260) \\ \delta_{cc} &= 349{,}000/EI \end{aligned}$$

There being no deflection of the supports, the equations of the form of 14 : 1 will be

$$\begin{aligned} \delta_{bo} + R_b\delta_{bb} + R_c\delta_{bc} &= 0 \\ \delta_{co} + R_b\delta_{cb} + R_c\delta_{cc} &= 0. \end{aligned}$$

Substituting the numerical values and multiplying all terms by $EI/1000$

$$\begin{aligned} -537{,}128 + 328\,R_b + 321\,R_c &= 0 \\ -558{,}032 + 321\,R_b + 349\,R_c &= 0 \end{aligned}$$

Whence $R_b = +727.5$ lb. and $R_c = +929.9$ lb.

Considering the included determinate structure under the previously known external loads and the reactions just computed acting together, and taking moments about D.

$$R_a \times 260 + 727.5 \times 160 + 929.9 \times 110 - 10 \times 305 \times 132.5 = 0$$

Whence $\quad R_a = +713.2$ lb.

and $\quad R_d = 3050 - (713.2 + 727.5 + 929.9) = 679.4$ lb.

The reactions having been obtained in this manner, the shears and bending moments can be computed in the usual manner, for the entire beam.

14 : 6. Deflection the First Derivative of the Internal Work — In a truss the internal work done when a system of external loads is gradually applied without exceeding the elastic limit of the material in any member is $W = \Sigma P^2L/2\,AE$, as was shown in Art. 13 : 2. If s is the load in any member due to a load of unity acting at some truss joint, a, X_a the external load on the truss at a in the direction of the unit load, and F_o that part of P due to other loads than X_a, we may write

$$P = F_o + sX_a \tag{a}$$

Then

$$W = \sum \frac{(F_o + sX_a)^2 L}{2\,AE} \tag{b}$$

and

$$\frac{\partial W}{\partial X_a} = \sum \frac{2(F_o + sX_a)sL}{2\,AE} = \sum \frac{sPL}{AE} \tag{c}$$

The right side of equation (c) is the expression for the deflection of a truss used in the Method of Work. Thus it can be seen that the deflection of any joint of a truss is equal to the partial derivative of the internal work caused by the application of the external loads, with respect to an external load acting at the joint in the direction of the desired deflection. This will be true whether the actual external load at the joint is finite or zero.

By the application of similar reasoning to the formula $W = \int_0^L \frac{M^2\,dx}{2\,EI}$ which represents the work done in bending, it can be shown that this principle also applies to beams subjected to bending only. The principle does not apply to beams subjected to combined axial and transverse loads, since the axial load and the bending in each member cannot be divided into two parts, one independent of the load at the point the deflection of which is desired and the other equal to the effect of that load if acting on the structure by itself.

14 : 7. Method of Least Work for External Forces — Suppose a beam or truss to be statically indeterminate with respect to the outer forces, and it is desired to compute the magnitudes of the reactions

under a specified loading, assuming no deflection of the supports. The statically indeterminate beam or truss may be treated as if it were an included statically determinate beam or truss subjected to both the known external loads and the unknown redundant reactions. The deflection of each redundant support point would be equal to the partial derivative of the work done in loading the structure, including that done by the unknown reactions, with respect to the unknown reaction at that point. By hypothesis, it would also be equal to zero. Expressed mathematically.

$$\frac{\partial W}{\partial R_a} = 0; \qquad \frac{\partial W}{\partial R_b} = 0; \ . \ . \ . \text{ etc.} \qquad 14:4$$

where W is the total work, and R_a, R_b, . . . the redundant reactions.

The values of the redundant reactions can therefore be found by writing the expression for the internal work done in loading the structure in terms of the known loads and unknown redundant reactions, differentiating this expression with respect to each of the unknowns, setting each of these partial derivatives equal to zero, and solving the resulting simultaneous equations. As W is a function of the independent variables R_a, R_b, . . . the values of the variables which will satisfy equations 14 : 4 are those which will give a minimum value of W consistent with the conditions of equilibrium, thus giving this method its name of the Method of Least Work.

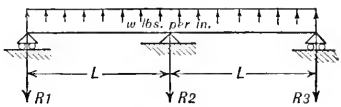


FIG. 14 : 2

The application of this method of determining reactions to a simple case is illustrated in the following numerical example.

Suppose it is desired to determine the reactions for a beam of constant section subjected to a uniformly distributed load, w lb. per in., and resting on three supports equally spaced as shown in Fig. 14 : 2. The first step is to write the equations of equilibrium

$$\Sigma V = 0; \qquad R_1 + R_2 + R_3 + 2\,wL = 0$$
$$\Sigma M = 0; \qquad 2\,R_1L + R_2L + 2\,wL^2 = 0$$

From these equations we can get $R_1 = R_3 = -wL - 0.5\,R_2$, thus expressing all three reactions in terms of the unknown R_2. The structure being symmetrical, the work done in both spans is the same and we can write for the internal work done

$$W = 2\int_0^L \frac{M^2\,dx}{2\,EI} = 2\int_0^L \frac{(R_1x + 0.5\,wx^2)^2\,dx}{2\,EI}$$
$$= \frac{1}{EI}\left(\frac{R_1{}^2L^3}{3} + \frac{R_1wL^4}{4} + \frac{w^2L^5}{20}\right)$$

Setting the derivative of this expression with respect to the unknown R_2 equal to zero, since $\dfrac{dR_1}{dR_2} = -\frac{1}{2}$, we may write

$$\frac{2\,R_1^2L^3}{3}\left(-\tfrac{1}{2}\right) + \frac{wL^4}{4}\left(-\tfrac{1}{2}\right) = 0$$

whence $R_1 = -3\,wL/8$. This is the same value that would have been obtained by the use of either the Three-moment Equation or Wilson's Method. The value of R_1 having been obtained, it is a simple matter to determine R_2 and R_3 from the equations of equilibrium.

14 : 8. Application of Least Work to Internal Forces — If a pin-jointed truss or framework, all the members of which are assumed to be subjected to axial loads only, contains redundant members, an included determinate structure can be obtained by cutting the redundant members. The net relative deflections of the pairs of cut faces should be zero when the included determinate structure is subjected to the known external loads and to loads on the cut faces equal to the internal forces in the respective members of the redundant structure. The relative deflection of each pair of cut faces will also be equal to the partial derivative, with respect to a load along the axis of the particular member cut, of the internal work done in loading the structure. Therefore we can express this phase of the principle of consistent deformations by the formulas

$$\frac{\partial W}{\partial X_a} = 0 \qquad \frac{\partial W}{\partial X_b} = 0 \qquad \frac{\partial W}{\partial X_c} = 0 \ldots \qquad 14:5$$

where X_a, X_b, X_c, etc., are the unknown loads in the redundant members A, B, C, etc.

It is easy to see that equations 14 : 5 are of the same type as equations 14 : 4, the difference being that the differentiations are with respect to the unknown forces in the redundant members instead of with respect to unknown reactions. Thus it can be seen that the Method of Least Work can be extended to apply to the computation of the unknown forces in the redundant members of a statically indeterminate structure built up of members subjected to axial loads only.

By similar reasoning it can be shown to apply also to members subjected to bending, in which the work done is equal to $\int M^2\,dx/2\,EI$.

14 : 9. General Method of Procedure — As stated in the derivation of the method, the system of stresses in the structure must be consistent with equilibrium. That is to say, the conditions of statics must be satisfied. The first step, therefore, is to write down all the independent equations representing relations between the forces in

the structure that can be obtained by applying the conditions of equilibrium, $\Sigma M = 0$, etc. The structure being indeterminate to the nth degree, there will be n more unknowns appearing in the equations thus obtained than there are equations.

There will also be several groups of n unknowns which might be simultaneously equal to zero without affecting the stability of the structure. One of these groups should be chosen and the values of all stresses in the structure determined in terms of the n unknowns. This is done by solving in the usual manner the simultaneous equations already obtained. The work done in all the members of the structure is now expressed in terms of the n unknowns and differentiated with respect to each unknown in turn, all unknowns except the one with respect to which the differentiation is being performed being treated as constants. Setting each of the n partial derivatives equal to zero produces n new independent equations which can be solved in the usual manner. All the unknowns can then be evaluated.

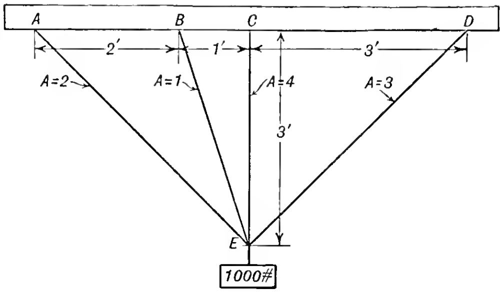


FIG. 14 : 3

This method of solution will be illustrated by a simple example.

Figure 14 : 3 shows a weight of 1000 lb. suspended from four wires, AE, BE, CE, and DE, the cross-sectional areas of which are in the ratios, 2, 1, 4, 3, respectively. It is desired to find the load in each wire.

As all four wires meet at E, lie in the same plane, and carry only axial load, there are four unknowns but only two available equations of equilibrium. There are thus two redundancies, or the structure is statically indeterminate in the second degree. It can also be seen that any two of the wires might be considered as the included determinate structure, since any other pair might be removed and the structure would still retain its stability. (This is not strictly true if wires are used, but would be if members capable of carrying compression were substituted.)

Writing down the two available equations of equilibrium, we have,

$$\Sigma V = 0 \qquad V_a + V_b + V_c + V_d - 1000 = 0 \tag{a}$$

$$\Sigma H = 0 \qquad H_a + H_b - H_c - H_d = 0 \tag{b}$$

where V_a and H_a are the vertical and horizontal components of the load in AE, and so on for the other members.

Expressing these equations in terms of the direct loads P_a, P_b, etc.,

$$0.707\ P_a + 0.949\ P_b + 1.000\ P_c + 0.707\ P_d = 1000 \qquad \text{(c)}$$
$$0.707\ P_a + 0.316\ P_b \qquad\qquad - 0.707\ P_d = 0 \qquad \text{(d)}$$

From (d),
$$P_a = P_d - 0.447\ P_b$$

Substituting this value of P_a in (c) and solving for P_c,

$$P_c = 1000 - 1.414\ P_d - 0.633\ P_b.$$

Knowing the four forces in terms of two of them, the expression for the internal work done can be written. For the structure in question, this expression will be

$$W = \frac{P_a{}^2 L_a}{2\ A_a E} + \frac{P_b{}^2 L_b}{2\ A_b E} + \frac{P_c{}^2 L_c}{2\ A_c E} + \frac{P_d{}^2 L_d}{2\ A_d E}$$
$$= \frac{4.2426\ P_a{}^2}{2 \times 2\ A_b E} + \frac{3.1623\ P_b{}^2}{2\ A_b E} + \frac{3.0000\ P_c{}^2}{2 \times 4\ A_b E} + \frac{4.2426\ P_d{}^2}{2 \times 3\ A_b E}$$
$$= \frac{1}{A_b E}(1.0607\ P_a{}^2 + 1.5812\ P_b{}^2 + 0.3750\ P_c{}^2 + 0.7071\ P_d{}^2) \qquad \text{(e)}$$

Substituting the values of P_a and P_c in terms of P_b and P_d,

$$W = \frac{1}{A_b E}[1.0607\ (P_d - 0.447\ P_b)^2 + 1.5812\ P_b{}^2 + 0.3750$$
$$(1000 - 1.414\ P_d - 0.633\ P_b)^2 + 0.7071\ P_d{}^2] \quad \text{(f)}$$

Differentiating W with respect to P_b and P_d in turn and setting the partial derivatives equal to zero we obtain

$$\frac{\partial W}{\partial P_b} = 0 = \frac{1}{A_b E}[1.0607 \times 2\ (-0.447)\ (P_d - 0.447\ P_b) + 2 \times 1.5812\ P_b$$
$$+ 0.3750 \times 2\ (-0.633)\ (1000 - 1.414\ P_d - 0.633\ P_b)] \qquad \text{(g)}$$

$$\frac{\partial W}{\partial P_d} = 0 = \frac{1}{A_b E}[1.0607 \times 2\ (P_d - 0.447\ P_b) + 0.3750 \times 2\ (-1.414)$$
$$(1000 - 1.414\ P_d - 0.633\ P_b) + 2 \times 0.7071\ P_d] \qquad \text{(h)}$$

Simplifying equations (g) and (h)

$$-0.4744\ P_d + 0.2122\ P_b + 1.5812\ P_b - 237.2 + 0.3355\ P_d$$
$$+ 0.1500\ P_b = 0 \qquad \text{(g')}$$
$$1.0607\ P_d - 0.4744\ P_b - 530.3 + 0.7500\ P_d + 0.3355\ P_b$$
$$+ 0.7071\ P_d = 0 \qquad \text{(h')}$$
$$1.9434\ P_b - 0.1189\ P_d = 237.2$$
$$-0.1389\ P_b + 2.5178\ P_d = 530.3$$
$$P_b - 0.0715\ P_d = 122.1$$
$$-P_b + 18.1267\ P_d = 3818.2$$
$$18.0552\ P_d = 3940.3$$
$$P_d = 218.2 \text{ lb.}$$
$$P_b = 137.7 \text{ lb.}$$

Substituting these values of P_b and P_d in the expressions for P_a and P_c,

$$P_a = 218.2 - 61.6 = 156.6 \text{ lb.}$$
$$P_c = 1000 - 87.1 - 308.6 = 604.3 \text{ lb.}$$

These values should be checked by substituting them in the original equations (c) and (d) which become

$$110.7 + 130.6 + 604.3 + 154.3 - 1000 = -0.1 \text{ instead of zero.}$$
$$110.7 + 43.5 - 154.3 = -0.1 \text{ instead of zero.}$$

The check is not exact but is close enough for any practical purposes. Such a check is always advisable as it shows that the system of forces found satisfies the conditions of statics. It is not, however, a positive check because any pair of values for P_b and P_d, if substituted in the expressions for P_a and P_c, would give a set of values that would satisfy it. Great care should therefore be taken that the expression for the internal work is correct and that no errors are made in the differentiation or in solving the equations obtained by that operation.

14 : 10. First Simplified Method — The method of solution illustrated above becomes excessively tedious and liable to error if there are many members in the structure. A simplification that is extremely useful for a moderate number of members subjected to axial loads and even more useful when bending is present is the application of the relation

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$

If this relationship is borne in mind it is not necessary to write the work expression exclusively in terms of the unknowns with respect to which it is to be differentiated, as in equation (f), but the differentiations can be made at the stage of equation (e). Using this short-cut, the computations down to and including equation (e) would be the same as before.

$$W = \frac{1}{A_bE}(1.0607\, P_a^2 + 1.5812\, P_b^2 + 0.3750\, P_c^2 + 0.7071\, P_d^2)$$

Differentiating with respect to P_b and P_d

$$\frac{\partial W}{\partial P_b} = \frac{1}{A_bE}\left[2 \times 1.0607\, P_a \frac{\partial P_a}{\partial P_b} + 2 \times 1.5812\, P_b + 2 \times 0.3750\, P_c \frac{\partial P_c}{\partial P_b}\right] = 0 \qquad \text{(g}''\text{)}$$

$$\frac{\partial W}{\partial P_d} = \frac{1}{A_bE}\left[2 \times 1.0607\, P_a \frac{\partial P_a}{\partial P_d} + 2 \times 0.3750\, P_c \frac{\partial P_c}{\partial P_b} + 2 \times 0.7071\, P_d\right] = 0 \qquad \text{(h}''\text{)}$$

Eliminating the term $2/(A_bE)$ which is common to both equations, and substituting the values of P_a and P_c and their derivatives with respect to P_b and P_d, we obtain from (g'') and (h'')

$$1.0607\,(-0.447)\,(P_d - 0.447\,P_b) + 1.5812\,P_b + 0.3750\,(-0.633)\,(1000 - 1.414\,P_d - 0.633\,P_b) = 0 \qquad (g''')$$

$$1.0607\,(P_d - 0.447\,P_b) + 0.3750\,(-1.414)\,(1000 - 1.414\,P_d - 0.633\,P_b) + 0.7071\,P_d = 0 \qquad (h''')$$

These equations are evidently identical with (g) and (h) and the remaining computations would be the same as before. In this illustrative case, this method of obtaining the partial derivatives of the internal work is no easier than the one first employed, but experience has shown that it makes the work much less tedious and much easier to check when the structure is more complex.

14 : 11. Second Simplified Method — When the structure is one such that the loads in all members are axial, the derivatives of the internal work with respect to the loads in the redundant members can be obtained from general formulas that make it unnecessary to write down the expression for the total work done. Using the nomenclature of Art. 14 : 4,

$$P = P_0 + X_aP_a + X_bP_b + \cdots \qquad 14:1$$

The internal work done will be

$$W = \sum \frac{(P_0 + X_aP_a + X_bP_b + \cdots)^2L}{2\,AE}$$

Differentiating and placing the partial derivatives equal to zero

$$\frac{\partial W}{\partial X_a} = \sum \frac{(P_0 + X_aP_a + X_bP_b + \cdots)P_aL}{AE} = 0$$

$$\frac{\partial W}{\partial X_b} = \sum \frac{(P_0 + X_aP_a + X_bP_b + \cdots)P_bL}{AE} = 0$$

Substituting Q for L/AE

$$\begin{aligned}\partial W/\partial X_a &= \Sigma P_0P_aQ + X_a\Sigma P_a^2Q + X_b\Sigma P_aP_bQ + \cdots = 0\\ \partial W/\partial X_b &= \Sigma P_0P_bQ + X_a\Sigma P_aP_bQ + X_b\Sigma P_b^2Q + \cdots = 0\end{aligned} \qquad 14:6$$

These equations can be written directly for any number of redundant members.

In applying this method to the structure shown in Fig. 14 : 3, the

first step is to determine which of the members shall be considered redundant. Any two may be chosen, and as they illustrate the problem best we will assume CE and DE to be redundant while AE and BE form the included determinate structure. Next determine the loads in the included determinate structure caused by the load of 1000 lb. and by unit tensions in the redundant members.

Under the load of 1000 lb.

$$\begin{aligned}
\Sigma V = 0 \qquad & 0.707\,P_a + 0.949\,P_b = 1000 \\
\Sigma H = 0 \qquad & 0.707\,P_a + 0.316\,P_b = 0 \\
& P_a + 1.342\,P_b = 1414.2 \\
& -P_a - 0.447\,P_b = 0 \\
& 0.895\,P_b = 1414.2 \\
& P_b = +1581 \text{ lb. } P_a = -707 \text{ lb.}
\end{aligned}$$

Under a load of 1 lb. tension in CE

$$P_a = +0.707 \text{ lb. } P_b = -1.581 \text{ lb.}$$

Under a load of 1 lb. tension in DE

$$\begin{aligned}
\Sigma V = 0 \qquad & 0.707\,P_a + 0.949\,P_b = -0.707 \\
\Sigma H = 0 \qquad & 0.707\,P_a + 0.316\,P_b = +0.707 \\
& +1.265\,P_b = -1.414 \\
& P_b = -2.236 \text{ lb. } P_a = +2.000 \text{ lb.}
\end{aligned}$$

The values of Q are then obtained for all four members. In this case, the areas being given in terms of A_b, the area of member BE, and E being assumed a constant, it will save labor if Q is expressed in terms of $1/A_bE$. Then

$$Q_a = 4.2426/2 = 2.1213 \qquad Q_b = 3.1623$$
$$Q_c = 3.000/4 = 0.7500 \qquad Q_d = 4.2426/3 = 1.4142$$

These quantities and the computations of the terms to be substituted in the equations 14 : 6 are given in Table 14 : 1.

TABLE 14 : 1

Member	Q	P_0	P_c	P_d	P_0P_cQ	P_0P_dQ	P_cP_dQ	P_c^2Q	P_d^2Q
AE	2.1213	−707	+0.707	+2.000	−1060	−3000	+3.000	+1.060	+8.485
BE	3.1623	+1581	−1.581	−2.236	−7904	−11179	+11.179	+7.904	+15.811
CE	0.7500	0	+1.000	0	0	0	0	+0.750	0
DE	1.4142	0	0	+1.000	0	0	0	0	+1.414
Sum					−8964	−14179	+14.179	+9.714	+25.710

Substituting in equations 14 : 6

$$
\begin{aligned}
-8964 + 9.714\,X_c + 14.179\,X_d &= 0 \\
-14179 + 14.179\,X_c + 25.710\,X_d &= 0 \\
X_c + 1.460\,X_d &= +922.8 \\
-X_c - 1.813\,X_d &= -1000.0 \\
-0.354\,X_d &= -77.2 \\
\text{Load in } DE = X_d &= +218.2 \text{ lb.} \\
\text{Load in } CE = X_c &= +604.3 \text{ lb.}
\end{aligned}
$$

$$\text{Load in } AE = -707 + 0.707 \times 604.3 + 2.000 \times 218.2 = +156.7 \text{ lb.}$$

$$\text{Load in } BE = +1581 - 1.581 \times 604.3 - 2.236 \times 218.2 = +137.6 \text{ lb.}$$

It should be noticed that these values check those obtained from the first computation within one-tenth of a pound.

This method is much more tedious than the other two for the structure used as an illustration, but is much easier when used in the solution of more complex structures. Its one defect is that it cannot be used for structures including members subjected to bending unless the formulas are modified, and in many cases of that kind the modifications involved are too difficult to be practicable.

14 : 12. Treatment of Work Due to Bending — When there is bending in any of the members of a structure to which either of the first two methods is applied, the work done in bending must be included in the expression for the total work in the structure. This is provided for by the formula

$$W = \int_0^L \frac{M^2\,dx}{2\,EI} \qquad 14:7$$

where W = work done in bending and the rest of the nomenclature is the same as in formula 13 : 3 of Art. 13 : 4.

An example of the use of this formula has already been given in Art. 14 : 7.

If M/EI cannot be represented as a continuous function of x, the members must be assumed to be divided into a group of members along each of which M^2/EI can be represented by a single function of x. In such cases the integrations are made between the ends of each of these members, in the same general manner as when the Method of Work is used to determine deflections.

If, in spite of the error involved, the Method of Least Work is applied to members subjected to combined bending and axial load, the work done in each member will be

$$W = \frac{P^2L}{2\,AE} + \int_0^L \frac{M^2\,dx}{2\,EI} \qquad 14:8$$

When the bending moment in any member is a constant, the work due to bending in that member will be $M^2L/2\,EI = M^2q/2$ if $q = L/EI$. For this case the formulas 14 : 6 can be modified by adding M terms similar to the P terms shown. Thus we would have

$$\frac{\partial W}{\partial X_a} = 0 = \Sigma(P_0P_aQ + M_0M_aq) + X_a\Sigma(P_a{}^2Q + M_a{}^2q) + X_b\Sigma(P_aP_bQ + M_aM_bq) + \cdots \qquad 14:9$$

and a similarly modified expression for $\partial W/\partial X_b$.

For members in which the bending moment varies, this device is not valid. It will, however, often be practicable to obtain expressions for the partial derivatives of the work done by the varying bending moments and add these to the partial derivatives of the work done by the axial loads and constant bending moments as found from formulas of the type of 14 : 8 to obtain the work equations desired.

A special case of varying bending moment that is often encountered is that in which the variation is uniform, there being no transverse load on the portion of the member being considered. For this case

$$\int_0^L \frac{M^2\,dx}{2\,EI} = \frac{L}{2\,EI}\left(M_1{}^2 \pm LM_1S + \frac{L^2S^2}{3}\right) \qquad 14:10$$

where M_1 is the bending moment at one end and S is the shear on the member. (There being no transverse load, this is a constant.)

The plus sign in the parenthesis is used when the shear tends to increase the bending moment as x increases, the minus sign when it tends to decrease the moment.

Formula 14 : 10 can be applied to a member with concentrated side loads by dividing the member into sections between adjacent loads and applying it to each section. It cannot be applied to sections subjected to a distributed side load. In such cases the value of M expressed in terms of x must be substituted in formula 14 : 7.

For a simply supported beam without overhangs and subjected to a uniform load the internal work done is $w^2L^5/120\,EI$.

Though the internal work done by the axial load may be added to that due to bending to obtain the total work in a structure, the bending load cannot be divided into portions (such as a constant bending, a uniformly varying bending, and a bending due to a uniformly distributed load), the work done by each of these portions computed, and the results added to obtain the total work in bending, because $(A + B)^2 = A^2 + 2\,AB + B^2$ *not* $A^2 + B^2$.

The application of the method of least work to a case involving members subjected to combined bending and compression can be

illustrated by a numerical example. Figure 14 : 4 shows a structure composed of two tubes OA and OB connected to a horizontal member AB and subjected to an upward load of 1200 lb. acting 2 in. to the right of O. The problem is to determine the loads in OA and OB, neglecting the work done in AB, but assuming that the joints at A and B are capable of transmitting bending moment. This is a simplified case of the general chassis analysis problem. The tubes are both of 1-in. outer diameter and 0.035-in. wall thickness, and the material is steel.

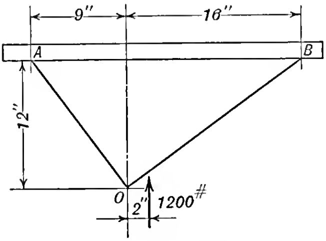


FIG. 14 : 4

There are seven external forces acting on the structure: the vertical load of 1200 lb., the axial loads P_a at A and P_b at B, the shear loads S_a at A and S_b at B, and the moments M_a at A and M_b at B. Figure 14 : 5 shows the forces at A and B in the directions in which they are assumed positive.

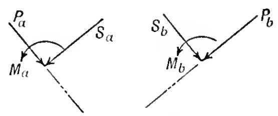


FIG. 14 : 5

Applying the conditions of equilibrium to the structure we obtain three equations:

$$\begin{aligned}
\Sigma V &= 0;\ 0.8\,P_a + 0.6\,S_a + 0.6\,P_b + 0.8\,S_b = 1200 \\
\Sigma H &= 0;\ 0.6\,P_a - 0.8\,S_a - 0.8\,P_b + 0.6\,S_b = 0 \\
\Sigma M &= 0;\ 15.0\,S_a - 20.0\,S_b + M_a + M_b = -2400
\end{aligned}$$

Whence

$$\begin{aligned}
P_b &= -S_a + 720 \\
S_b &= -P_a + 960 \\
M_b &= -M_a - 15\,S_a - 20\,P_a + 16{,}800
\end{aligned}$$

The expression for total work is

$$W = \sum\left[\frac{P^2L}{2\,AE} + \frac{L}{2\,EI}\left(M^2 + LMS + \frac{L^2S^2}{3}\right)\right] \qquad 14 : 10a$$

Substituting the numerical values for the specific problem

$$\begin{aligned}
EW &= \frac{15\,P_a^2}{2 \times 0.10611} + \frac{15}{2 \times 0.01237}\left[M_a^2 + 15\,M_aS_a + \frac{225\,S_a^2}{3}\right] \\
&\quad + \frac{20\,P_b^2}{2 \times 0.10611} + \frac{20}{2 \times 0.01237}\left[M_b^2 - 20\,M_bS_b + \frac{400\,S_b^2}{3}\right] \\
EW &= 70.68\,P_a^2 + 606.30\,(M_a^2 + 15\,M_aS_a + 75\,S_a^2) + 94.24\,P_b^2 \\
&\quad + 808.40\,(M_b^2 - 20\,M_bS_b + 133.33\,S_b^2)
\end{aligned}$$

As the values of the unknowns with the subscript b have already been determined in terms of those with the subscript a, we will differentiate the above work expression with respect to each of the latter and set the partial derivatives equal to zero, thus obtaining the three additional equations needed for the solution of the problem. It may be noted that in this case the expression being differentiated is that for EW rather than that for W, but as E is a constant and, as the partial derivatives are all to be assumed equal to zero, the resulting equations are the same as if E had been kept on the right side and cancelled out after the differentiations had been made.

$$\frac{\partial EW}{\partial P_a} = 2 \times 70.68\, P_a + 808.40\,(-20 \times 2\, M_b + 20 \times 20\, S_b + 20\, M_b - 2 \times 133.33\, S_b) = 0$$

$$\frac{\partial EW}{\partial S_a} = 606.30\,(15\, M_a + 2 \times 75\, S_a) - 2 \times 94.24\, P_b + 808.40\,(-2 \times 15\, M_b + 20 \times 15\, S_b) = 0$$

$$\frac{\partial EW}{\partial M_a} = 606.30\,(2\, M_a + 15\, S_a) + 808.40\,(-2\, M_b + 20\, S_b) = 0$$

Whence

$$70.68\, P_a + 53893\, S_b - 8084.0\, M_b = 0$$

$$90945.0\, S_a + 9094.5\, M_a - 188.48\, P_b + 242{,}520\, S_b - 24{,}252.0\, M_b = 0$$

$$9094.5\, S_a + 1212.6\, M_a + 16{,}168\, S_b - 1{,}616.8\, M_b = 0$$

Due to the method of differentiation used, the unknowns with the subscript b have reappeared in these "least-work equations." One method of attack would be to combine these least-work equations with the three original equations of equilibrium and solve the resulting group of six simultaneous equations. A somewhat simpler procedure would be to substitute the values of P_b, S_b, and M_b in terms of P_a, S_a, and M_a already computed into the least-work equations, thus obtaining a group of only three simultaneous equations with three unknowns. In this particular example, it will be seen that as P_a and P_b appear in the least work equations only once, and M_a only twice, the work would be simplified if the values of those three unknowns in terms of the other three were substituted. This is particularly advantageous on account of the simple relations that can be used. Making these substitutions, the least work equations become:

$$\begin{aligned} 53{,}822.3\, S_b - 8{,}084.0\, M_b &= -67{,}852.8 \\ -45{,}284.0\, S_a + 424{,}410.0\, S_b - 33{,}346.5\, M_b &= +21{,}962{,}505.6 \\ -9{,}094.5\, S_a + 40{,}420.0\, S_b - 2{,}829.4\, M_b &= +2{,}910{,}240.0 \end{aligned}$$

Whence

$$S_a = -131.8 \text{ lb.}$$

$$S_b = 80.4 \text{ lb.}$$

$$M_b = 543.8 \text{ in.-lb.}$$

Substituting these values in the original equations of equilibrium,

$$P_a = 879.6 \text{ lb.}$$
$$P_b = 851.8 \text{ lb.}$$
$$M_a = 641.1 \text{ in.-lb.}$$

The minus sign of S_a indicates that it acts in the direction opposite to that shown in Fig. 14.5. The plus sign of the other quantities indicate that they act in the directions shown in that figure.

With the components of the reactions at A and B known, the bending moment and shear at any point in OA and OB can be computed by the methods already discussed.

The computations required in a case like this are usually very tedious, this example being a relatively simple one, and the method is often avoided for that reason. Nevertheless it is the most precise method available for many designs. The labor of computation varies with the number of redundant unknowns and is greatly decreased if certain of those unknowns can be given assumed values. In the above problem, for example, the work of computation would have been greatly decreased if the moments at A and B had been assumed equal to zero, i.e., that the joints at these locations were pinned.

The above example also illustrates two weaknesses of the Method of Least Work. Often, as in this case, the work done in the structure supporting that being investigated must be neglected either to reduce the labor of computation to a reasonable amount or because the design of that structure has not yet been made. This causes an error of unknown amount. The other weakness is that although the cross-sections of members at joints usually differ from those between joints it is customary to assume the members as of constant section and to extend from center line intersection to center line intersection.

It may also be noticed that the work done by the axial load was so small in comparison with that due to bending that the former could have been omitted from the total work expression without seriously affecting the results. In approximate and preliminary computations, much labor can often be saved if this practice is followed.

14 : 13. Initial Stresses in Trusses — If a statically determinate pin-jointed truss were subjected to no external loads there would be no internal stresses in its members other than those caused by the dead weight. This can be proved by trying to compute the stresses in any such truss. When external forces are absent, the equations of equilibrium will show that the internal forces are all zero.

A statically indeterminate pin-jointed truss, however, may have loads in its various members even though it is subjected to no external

loads. The nature of these loads can best be explained if we consider the indeterminate truss as built up by the addition of redundant members to an original determinate truss. If the lengths of the redundant members are just equal to the distances between the pins on the original truss to which they are connected, they can be inserted without subjecting the original truss or the redundant members to any loads. If any of the redundant members are not of just the right length, both they and the original truss must be deformed to permit their insertion. This will cause the redundant truss to be subjected to "initial stresses."

Suppose for example that the length of a redundant member AB is less than the distance between joints A and B of the original truss by an amount δ, which may be called the "initial deformation" for member AB. If one end of the redundant member is connected to joint B of the truss, the other end will fall short of joint A by the distance δ. Equal and opposite forces must therefore be applied to the free end of the redundant member and to joint A, of sufficient magnitude to give them a relative deflection equal to δ, before the connection at A can be made. After the connection is made the internal stresses produced in this operation will remain in the structure as initial stresses.

In the derivation of the formulas in Art. 14 : 8, the redundant members were assumed to have been cut. The loads in them were found by equating the sum of the relative deflections of the cut faces due to the known external loads and to the loads on the cut faces, to zero. This procedure is correct only when no initial deformations were necessary to insert the redundant members. If such initial deformations existed, it would be necessary to equate the sum of the relative deflections due to the known external loads and those due to the loads on the cut faces (the loads in the redundant members) to the various initial deformations. The fundamental equations for a truss with initial deformations and stresses are therefore

$$\frac{\partial W}{\partial X_a} = \delta_a \qquad \frac{\partial W}{\partial X_b} = \delta_b \qquad \frac{\partial W}{\partial X_c} = \delta_c \qquad 14:11$$

in which δ_a, δ_b, δ_c, etc., are the initial deformations for the redundant members in which the loads are X_a, X_b, X_c, etc., respectively.

The loads in the redundant members could be divided into two parts, the initial loads producing the deformations δ, and the subsequent loads induced by the known external loads. The initial loads for any given set of values of δ could be obtained from equations 14 : 11 if W were taken to be the internal work done in applying those loads only. The subsequent loads would then be obtainable from the equations 14 : 5. As the principle of super-position applies, however,

equations 14 : 11 can be used to determine the total loads in the redundant members, W representing the total work done, including both that done in assembling the structure and that done by the application of the known external loads.

It might be thought that if the initial deformations were known or assumed the initial stresses in the redundant members could be computed very simply by calculating the loads required to change the length of each redundant member an amount equal to the corresponding initial deformation. This would not give the correct result because δ is the relative deflection of the cut faces, or of the end of the member and the joint to which it is to be attached. Only a part of this deflection is due to change in length of the redundant member, the remainder being due to changes in length of members of the original truss and to changes in length of any other redundant members.

It should be evident from the foregoing discussion that the loads in the redundant members of an indeterminate truss can be computed by the Formulas 14 : 6 of Art. 14 : 11 if the known or assumed values of δ are substituted for the zeros shown in those formulas. If the initial deformations are due to the redundant members being too short, they will cause tensions in those members and the values of δ should be considered positive; and vice versa.

When the effects of initial deformations are being determined, the computer cannot use values of Q in terms of EA or E as was done in the numerical example of Art. 14 : 11 unless he uses values of δEA or δE in place of the quantity δ in formula 14 : 11. Often, however, this work will be simplified by doing this in the manner illustrated by the numerical example in the next article.

14 : 14. Members Dropping Out of Action — Many trusses used in airplane work are statically indeterminate, but some of the redundant members are wires which cannot transmit compression. Wing trusses are usually of this type, being indeterminate if both lift and landing wires are considered. In this case, if the air loads act up, the landing wires, being unable to carry compression, go out of action, and the remaining members form a determinate truss. If there were no initial tensions in the wires, the landing wires would drop out of action at once, but usually initial stresses are introduced in rigging, so that the landing wires do not become inactive until a certain amount of external load has been imposed. Though it is not often necessary to compute the effect of the initial stresses in such cases, it is occasionally necessary to do so, and the method will be illustrated by an example.

Suppose the loads in the portion of fuselage truss shown in Fig. 14 : 6 are desired, first when there are no initial stresses, and second when

TABLE 14 : 2

Member	P_0	QE	P_1	P_2	P_3	P_0P_1QE	P_0P_2QE	P_0P_3QE	P_1^2QE	P_2^2QE	P_3^2QE	P_1P_2QE	P_1P_3QE	P_2P_3QE
AB	−5100	180	−0.6	0	0	+550,800	0	0	64.8	0	0	0	0	0
BC	−3600	180	0	−0.6	0	0	+388,800	0	0	64.8	0	0	0	0
CD	−2400	320	0	0	−0.8	0	0	+614,400	0	0	204.8	0	0	0
FG	+3600	180	−0.6	0	0	−388,800	0	0	64.8	0	0	0	0	0
GH	+2400	180	0	−0.6	0	0	−259,200	0	0	64.8	0	0	0	0
HK	+800	320	0	0	−0.8	0	0	−204,800	0	0	204.8	0	0	0
AF	−2400	240	−0.8	0	0	+460,800	0	0	153.6	0	0	0	0	0
BG	−2000	240	−0.8	−0.8	0	+384,000	+384,000	0	153.6	153.6	0	+153.6	0	0
CH	−1600	240	0	−0.8	−0.6	0	+307,200	+230,400	0	153.6	86.4	0	0	+153.6
DK	−1200	240	0	0	−0.6	0	0	+172,800	0	0	86.4	0	0	0
AG	0	1500	+1.0	0	0	0	0	0	1500.0	0	0	0	0	0
BF	+2500	1500	+1.0	0	0	+3,750,000	0	0	1500.0	0	0	0	0	0
BH	0	1500	0	+1.0	0	0	0	0	0	1500.0	0	0	0	0
CG	+2000	1500	0	+1.0	0	0	+3,000,000	0	0	1500.0	0	0	0	0
CK	0	2000	0	0	+1.0	0	0	0	0	0	2000.0	0	0	0
DH	+2000	2000	0	0	+1.0	0	0	+4,000,000	0	0	2000.0	0	0	0
Sum						−4,756,800	+3,820,800	+4,812,000	+3436.8	+3436.8	+4582.4	+153.6	0	+153.6

there are initial deformations of 0.1 in. for each of the redundant wires AG, BH, and CK. It will be assumed that all horizontal and vertical members are tubes with a sectional area of 0.100 sq. in. and all diagonals are wires of 0.020 sq. in. sectional area. The modulus of elasticity will be taken as 30,000,000 lb. per sq. in. The quantities shown in formulas 14 : 6 being needed for the solution of the problem, they are computed and tabulated in Table 14 : 2.

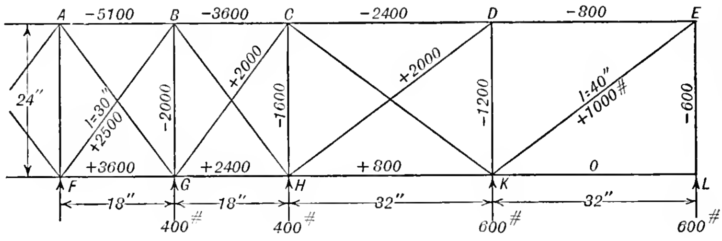


Fig. 14 : 6

As it is first desired to compute the loads in the redundant wires assuming no initial deformations, we have the equations,

$$
\begin{aligned}
\Sigma P_0P_aQE + X_a\Sigma P_a^2QE \quad + X_b\Sigma P_aP_bQE + X_c\Sigma P_aP_cQE &= 0\\
\Sigma P_0P_bQE + X_a\Sigma P_aP_bQE + X_b\Sigma P_b^2QE \quad + X_c\Sigma P_bP_cQE &= 0\\
\Sigma P_0P_cQE + X_a\Sigma P_aP_cQE + X_b\Sigma P_bP_cQE + X_c\Sigma P_c^2QE &= 0\\
4{,}756{,}800 + 3{,}436.8\,X_a + \quad 153.6\,X_b \qquad\qquad &= 0\\
3{,}820{,}800 + \quad 153.6\,X_a + 3{,}436.8\,X_b + 153.6\,X_c &= 0\\
4{,}812{,}800 \qquad\qquad + \quad 153.6\,X_b + 4{,}582.4\,X_c &= 0
\end{aligned}
$$

Whence

$$X_a = -1{,}339.1 \text{ lb.} \qquad X_b = -1{,}006.4 \text{ lb.} \qquad X_c = -1{,}016.6 \text{ lb.}$$

All of these are purely hypothetical loads as the members are wires and cannot carry compression. What these figures really indicate is that all three wires would be out of action, as would be expected since they were assumed to be subjected to no initial stress.

If each wire is assumed to have been originally stressed due to initial deformations of 0.1 in., the formulas would be

$$
\begin{aligned}
4{,}756{,}800 + 3{,}436.8\,X_a + \quad 153.6\,X_b \qquad\qquad &= 3{,}000{,}000\\
3{,}820{,}800 + \quad 153.6\,X_a + 3{,}436.8\,X_b + \quad 153.6\,X_c &= 3{,}000{,}000\\
4{,}812{,}800 \qquad\qquad + \quad 153.6\,X_b + 4{,}582.4\,X_c &= 3{,}000{,}000
\end{aligned}
$$

Whence

$$X_a = -502 \text{ lb.} \qquad X_b = -199 \text{ lb.} \qquad X_c = -389 \text{ lb.}$$

Comparison of the results shows that the redundant members would not have dropped out so soon in the second as in the first case. It may be necessary, however, to determine at what percentages of the total external loads the various redundant wires would go out of action. If k is the percentage of the external load in action at any moment, the above equations can be written

$$\begin{aligned} 4{,}756{,}800\,k + 3{,}436.8\,X_a + 153.6\,X_b &= 3{,}000{,}000 \\ 3{,}820{,}800\,k + 153.6\,X_a + 3{,}436.8\,X_b + 153.6\,X_c &= 3{,}000{,}000 \\ 4{,}812{,}800\,k + 153.6\,X_b + 4{,}582.4\,X_c &= 3{,}000{,}000 \end{aligned}$$

Whence

$$\begin{aligned} X_a &= 836.8 - 1{,}339.1\,k \\ X_b &= 807.4 - 1{,}006.4\,k \\ X_c &= 627.6 - 1{,}016.6\,k \end{aligned}$$

in which the constant is the initial tension in the wire and the k term the effect of the external loads.

From the above values of X_a, it appears that AG would go out of action when $k = 836.8/1{,}339.1 = 0.625$. Similarly X_b would become zero indicating that member BH was on the point of becoming inactive when $k = 807.4/1{,}006.4 = 0.802$. The critical point for member CK would be that when $k = 627.6/1{,}016.6 = 0.617$. The lowest of these values is that for CK, showing that it would be the first of the three wires to go out of action.

When CK has gone out of action, the structure becomes indeterminate in only the second degree, and to determine stresses when k is a little larger than 0.617, the equations must be modified by dropping out all terms referring to CK. The equations then become

$$\begin{aligned} 4{,}756{,}800\,k + 3{,}436.8\,X_a + 153.6\,X_b &= 3{,}000{,}000 \\ 3{,}820{,}800\,k + 153.6\,X_a + 3{,}436.8\,X_b &= 3{,}000{,}000 \end{aligned}$$

Whence

$$X_a = 835.6 - 1{,}337.1\,k \text{ and } X_b = 835.2 - 1{,}051.2\,k$$

These expressions for X_a and X_b differ from those computed first, but give the same values when $k = 0.617$. They indicate that member AG would go out of action when $k = 835.6/1{,}337.1 = 0.625$ and that BH would drop out when $k = 835.2/1{,}051.2 = 0.795$.

When AG has gone out of action, BH is the only redundant wire remaining in action, and the load in it can be determined from

$$3{,}820{,}800\,k + 3{,}436.8\,X_b = 3{,}000{,}000$$

Whence

$$X_b = 872.9 - 1{,}111.7\,k.$$

This member would go out of action when $k = 872.9/1{,}111.7 = 0.802$.

It happens that in this case, the complete computations show that the redundant wires all go out of action at the percentages of full load indicated in the first computation. This is due to the fact that these values of k were computed to only three significant figures, and that there was not much spread between the values of k at which the wires went out of action. It could not be counted on to happen generally, though the error in assuming that the various wires would go out of action at the percentages of the total load indicated from the computations based on all redundant wires being in action is not likely to be large in any practical case. It is much less likely to be serious than errors in the assumed values of the initial deformations.

14 : 15. Value of the Methods — Although the computations involved in analyzing indeterminate structures by the methods discussed in this chapter are often very tedious, and it is hard to catch arithmetical errors except by independent checking, the computations are not at all difficult, unless the number of redundant quantities becomes excessive. In most airplane structures there are a large number of redundancies that would have to be considered if an exact analysis were attempted. Most of them have a negligible effect on the stress systems, however, and should be neglected. This is the more justified since an attempt to allow for all redundancies would not suffice for a precise analysis owing to inherent errors in the method. Among these inherent errors are: the basic formulas pre-suppose that the principle of superposition is applicable, though this is not true with respect to members subjected to combined axial and bending loads; the exact amounts of all the initial deformations in the structure can never be known and will usually differ between different articles of the same design; the cross-sections and moduli of elasticity of the members are never just what they are assumed to be in the computations; in practice it would be impossible to take account of all variations in cross-section affecting only a small portion of the length of the member affected; it is impossible to determine the work done in complex connecting fittings with their complex and unknown stress distributions; and the supports are never fully rigid and the degree of rigidity attained by them can seldom be known. Finally, even though all of the above defects could be overcome, the external loads are never known with complete accuracy, and it is useless to attempt to compute the internal loads to a greater degree of accuracy than that to which the external loads are known.

The existence of these defects in the methods for analyzing redundant structures does not mean that they are useless or that they should be superseded by arbitrary assumptions. It means only that they should be used with judgment, and their limitations recognized.

As a designer becomes more practiced in the use of these methods of analysis he will learn what redundancies can safely be left out of account in familiar types of structure. He will also learn how to avoid much work of detailed computation by neglecting such of the terms in the work expression as will have little influence on the final results. In addition he will be able to make much more reasonable arbitrary assumptions as to the magnitudes of the redundant unknowns than if he lacked this experience.

14 : 16. Arrangement of Computations — The labor involved in analyzing complex redundant structures can usually be greatly reduced if the computations are carried out in a systematic form. The suggested practice outlined below for organizing the computations by the Second Simplified Method of Least Work has been found by experience to be very helpful. It is taken from Air Corps Information Circular 495, "Application of the Method of Least Work to Redundant Structures," by C. J. Rowe, in which its use is illustrated by a numerical example.

(1) Show completely dimensioned line drawings of the framework to be analyzed.

(2) Tabulate the physical properties and directional components of the members in the framework and any others which will affect the stresses in these members.

(3) Make line drawings of the framework, showing the point of application and the magnitude and direction of all applied loads affecting the framework. Give the nomenclature to be used.

(4) Very carefully select the redundant members, trying to choose those which may go out of action.

(5) Write the work equations which will be equal in number to the number of redundant members.

(6) Make a line drawing to show the statically determinate structure obtained by temporarily removing the redundant members, and solve this structure for the stresses due to applied loads. These values should be tabulated under the heading *Po*. Show, in the right or left margin of the page, which member is being solved for by the equation on that line.

(7) Show, by line drawings, that part of the framework affected by a unit load in each of the redundant members and solve for the stresses due to this load. Tabulate these values under the heading P with subscripts designating the various redundant members.

(8) Tabulate the foregoing values and solve for the numerical values required in the work equations.

(9) Before solving the work equations, arrange the various terms so that those representing members which may be found to be out of action will appear either at the beginning or the end of the equations.

(10) Arrange the various equations in a series so that those composed

entirely of functions of the loads in the doubtful members will appear at the beginning or end of the series.

(11) Tabulate and solve the equations according to some method such as is described below.

(12) Multiply the stresses in the various members, due to unit loads in the redundant members by the actual loads obtained in these members, and take the algebraic sum of these stresses together with those under the heading *Po* and obtain the final stress in each member of the framework.

In the same report from which the above recommendations are quoted, Mr. Rowe gives the following suggestions regarding the solution of a number of simultaneous linear equations.

(a) Write the equations to be solved in tabular form, one line for each equation, and one column for each variable. It will save labor if the symbol for the variable (x, y, etc.) is placed only at the head of the column, the coefficients alone appearing in the lines below. If one of the variables does not appear in one of the equations, the space at the intersection of the corresponding line and column should be left blank or a zero inserted at the option of the computer.

(b) In a column at one side of the table of equations each should be numbered or lettered for reference.

(c) Divide each equation by plus or minus the coefficient of the variable in the first column and write down the resulting equations directly below the first set. The same columns should be used for each variable. The plus or minus sign should be used with these divisors in such a manner that the coefficients of the variables in the first column will be alternately plus one and minus one. The equations of this second set should be referenced by the same numbers or letters as those of the first set.

(d) Add adjacent equations of the second set to obtain the equations of the third set, and write the latter directly below, using the same column for each variable. As the coefficients of the first variable are plus and minus unity, this variable will disappear. By this process both the number of variables and the number of independent equations will be reduced by one.

(e) Reference this third set of equations by a new set of numbers or letters, and show in the column where the referencing symbols appear, how the new equation was formed. Thus, if equations 1 and 2 were added to obtain equation 6, the note in the column of referencing symbols should read "$1 + 2 = 6$."

(f) Repeat the above operations until a numerical value is obtained for one of the unknown variables.

(g) Substitute this value in both of the equations of the set with two unknowns, in which the coefficients of the other variable appearing are plus and minus one, and solve for that variable. Consider the mean of the values obtained as the correct value. If these two values differ appreciably an error has been made that should be sought out and corrected.

(h) Repeat the above operation until all of the unknown variables have been evaluated.

After one of the variables has been evaluated, it is quicker to use only one of each set of equations in computing the values of the other variables, but the likelihood of error is greater. In this case, the values of all of the variables should be checked by substituting them in all of the equations of the first set to determine whether identities result. Exact identities may not be obtained due to the use of a limited number of significant figures, but if the deviation from an identity is large, an error has been made and the computations must be corrected.

14 : 17. Distribution of Load on Multi-Spar Wings — Another application of the principle of consistent deformations is involved in the determination of the load acting on each of the spars in a multi-spar wing. It is assumed in such investigations that the deflection of any cross-section of a wing may be broken up into a vertical deflection and a rotation. It is further assumed that there is no distortion of the cross-section so that points which lie on a straight line before the load is applied will continue to lie on a straight line after it is applied. These assumptions have been vindicated by tests on one large wing,[1] and, though tests on a single structure do not suffice to establish the absolute validity of such assumptions they do indicate the assumptions to be reasonable.

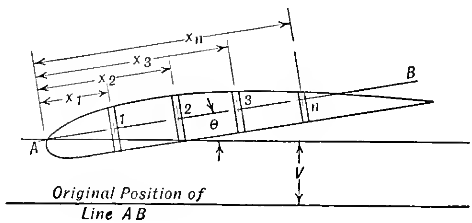


FIG. 14 : 7

Fig. 14 : 7 shows the cross-section of a wing in its position after a load is applied. Line AB has deflected vertically a distance v and rotated through an angle θ. Then the total deflection, δ_1, of point 1 is $v + \theta x_1$; and for point 2, $\delta_2 = v + \theta x_2$. Hence, $\dfrac{\delta_1}{\delta_2} = \dfrac{v + \theta x_1}{v + \theta x_2}$. We have seen, however, in Art. 13 : 1 that the deflection of any beam may be expressed in the form $\delta = \dfrac{CwL^3}{EI}$ where C is a constant which depends on the distribution of the load and the method of support. By rearranging this expression and writing $k = \dfrac{E}{CL^3}$ we get, $w = kI\delta$. Hence, if the deflection of a beam, its properties and the method of supporting it are known, the load causing the deflection may be computed, and if k

[1] Air Corps Information Circular 394, "The Distribution of Load Among the Spars in Multi-Spar Construction of Airplane Wings," by J. S. Newell.

is the same for several beams the loads applied to each to produce a given deflection may be obtained. For trussed beams, beams of different lengths or those having different variations in moment of inertia along the span, k would not be the same, but for the types of construction ordinarily encountered in multi-spar wings the variation in k for the different spars is not great. For a given structure k can therefore be assumed the same for all spars without appreciable error. Then:

$$\frac{\dfrac{w_1}{kI_1}}{\dfrac{w_2}{kI_2}} = \frac{v + \theta x_1}{v + \theta x_2}$$

Subtracting 1 from both sides, and simplifying,

$$\frac{\dfrac{w_1}{I_1} - \dfrac{w_2}{I_2}}{\dfrac{w_2}{I_2}} = \frac{\theta(x_1 - x_2)}{v + \theta x_2} \qquad 14:12$$

By a similar process,

$$\frac{\dfrac{w_3}{I_3} - \dfrac{w_2}{I_2}}{\dfrac{w_2}{I_2}} = \frac{\theta(x_3 - x_2)}{v + \theta x_2} \qquad 14:13$$

Dividing each side of equation 14 : 12 by the corresponding side of 14 : 13 and changing the signs of the terms in the numerators on each side of the resulting equation,

$$\frac{\dfrac{w_2}{I_2} - \dfrac{w_1}{I_1}}{\dfrac{w_3}{I_3} - \dfrac{w_2}{I_2}} = \frac{x_2 - x_1}{x_3 - x_2} \qquad 14:14$$

For the n th, $n - 1$st and $n - 2$nd spars the expression would be

$$\frac{\dfrac{w_{n-1}}{I_{n-1}} - \dfrac{w_{n-2}}{I_{n-2}}}{\dfrac{w_n}{I_n} - \dfrac{w_{n-1}}{I_{n-1}}} = \frac{x_{n-1} - x_{n-2}}{x_n - x_{n-1}} \qquad 14:15$$

For equilibrium the summation of the loads acting on the spars at any section must equal the load, F, on that section, hence,

$$w_1 + w_2 + \cdot\ \cdot\ \cdot\ w_n = F \qquad 14:16$$

and the center of pressure of the loads on the spars must correspond with the center of pressure of the load on the section. Then,

$$\frac{w_1x_1 + w_2x_2 + \cdot \cdot \cdot w_nx_n}{w_1 + w_2 + \cdot \cdot \cdot w_n} = c.p. \qquad 14:17$$

Formulas 14 : 15, 14 : 16, and 14 : 17 will furnish sufficient equations so that the values of w may be determined for any number of spars at a given cross-section. As in practically all solutions which involve the method of consistent deformations, a knowledge of the sizes of the spars is necessary before the formulas may be applied. Trial designs are therefore obligatory and it is necessary to investigate the distribution at several sections along a wing in order to obtain satisfactory loading curves on which to base the spar design.

When investigating a wing of the type in which the two chords of a spar do not lie in a single vertical plane it will be necessary to provide phantom chords vertically above, or below, the actual members, the actual area being equally divided between the existing member and its phantom. The computations may then be carried out as in the ordinary case.

The above formulas are identical with those derived by Prof. E. P. Warner of the Massachusetts Institute of Technology but they are obtained in a somewhat simpler manner. They also give results which agree with those of the Burgess Rational Method described in Information Circular 394 but involve much less laborious computations.

14 : 18. Illustrative Problem — Determine the distribution of load on a three-spar wing under a running load of 100 lb. per in. at a section where the spars are at 15, 40 and 65 per cent of the chord and have moments of inertia as follows: $I_1 = 86$, $I_2 = 117$ and $I_3 = 72$ in.^{4} The center of pressure is at 50 per cent of the chord from the leading edge in the Low Angle of Attack condition.

Substituting in Formulas 14 : 15, 14 : 16, and 14 : 17:

$$\frac{\frac{w_2}{117} - \frac{w_1}{86}}{\frac{w_3}{72} - \frac{w_2}{117}} = \frac{40 - 15}{65 - 40}$$

$$w_1 + w_2 + w_3 = 100$$

$$\frac{w_1 \times 15 + w_2 \times 40 + w_3 \times 65}{w_1 + w_2 + w_3} = 50$$

Solving these three equations, we get, $w_1 = 7.9$, $w_2 = 44.2$ and $w_3 = 47.9$ lb. per in.

By similar computations at other sections we can determine the

distribution among the spars and thus obtain curves showing the load on each spar, from which the bending moments and shears at various sections may be computed for use in checking the trial design. If the stresses obtained are too high the spar sections are modified, their new moments of inertia computed and the new distribution of load determined. The final design is thus obtained by a system of trial and error and, although considerable work of a more or less tedious nature is involved, the mathematical equations involved are simple and easily handled.

PROBLEMS

14 : 1. Assume wire CE to be removed from the airship power-car suspension shown in Fig. 14 : 3 and determine the stresses in the three remaining wires by the Method of Least Work described in Art. 14 : 11.

14 : 2. Compute the bending moment at the base of the rear fin spar shown in Fig. 4 : 4 under the following assumptions:

(*a*) Assume the brace wire which is in action to be rigid, *i.e.*, the work in it to be zero.

(*b*) Assume the brace wire to have a value of $EA = 200{,}000$ lb.

In both of the above cases use a 1 1/8–0.049 mild steel (S.A.E. 1025) tube for the fin spar and consider the work due to axial loads and primary bending moments only.

14 : 3. Determine the reactions for a truss dimensioned and loaded as in Fig. 13 : 12 but having a third support at point J. Use the Method of Least Work and compare your results with those obtained in Problem **13 : 2.**

14 : 4. An airplane wing has spars at 15, 30, 45 and 60 per cent of the chord. They are of rectangular cross-section and have the following dimensions: 1 in. by 5 in., 1 1/4 in. by 6 in., 1 in. by 5 in., and 3/4 in. by 4 in. If the center of pressure is at 28 per cent of the chord in the high angle of attack condition, how much load does each spar carry in per cent of the running load per inch of wing span?

CHAPTER XV

WEIGHT ESTIMATION

One of the most difficult steps in the development of a new airplane is the estimation of the weight of the completed design. It is also one of the most important steps, since errors in the preliminary estimation of weight practically always result in lower performance and decreased ability to carry pay load. The estimation of weight cannot be done mechanically, but must be tempered by sound judgment applied to a stock of reliable weight data. It is the purpose of this chapter to present as many of these data as possible in a convenient form for use in preliminary estimates, with some notes regarding the proper method of using the data.

15 : 1. Weight Estimates and Weight Computations — When complete drawings have been made for an airplane or any one of its parts, the corresponding weight can be computed with a precision of 1 or 2 per cent by the simple and obvious though tedious method of determining the volume or area of each part and, from a knowledge of the unit weights, computing the total weight of that part. The major causes of error in this type of computation are:

1. Deviations of the density of the materials used from the values assumed in the calculations.
2. Deviations of the dimensions of the article, as manufactured, from the drawings, such as those due to taking advantage of manufacturing tolerances.
3. Neglect of minor items such as rivet heads, varnish, etc.
4. Errors due to the difficulty of computing the exact volume or area of objects of irregular shape.

The effect of the first two of these sources of error is such that if several articles are constructed in accordance with the same drawings their actual weights will vary. For this reason, one company which keeps a very close check on its weights makes the practice, whenever possible, of weighing ten of each part constructed and using the average weight as the weight of the item. This is particularly advisable in the case of parts made up largely of wood, on account of the variations in density of the material, and of metal castings, for which the dimensional tolerances are relatively large.

Although the designer should compute the weight of each part as soon as its drawings have been completed, and should check that weight

against the actual weight as soon as the part has been made, he cannot get along with this type of weight computation alone. At the very start of the design process, before any drawings have been made, and at various intermediate stages, he must estimate the gross weight of the airplane and its major divisions. Any weight figure that is not based upon the detailed dimensions of the item in question should be called an estimated weight, and the process by which it is obtained, weight estimation. The term weight computation should be restricted to computations based upon detailed dimensions.

The greater the number of dimensions that have been determined, the more nearly the process of weight estimation approaches that of weight computation, but until a part has been completely detailed, allowances based on the judgment of the engineer must be made to care for the effect of the undetermined dimensions. The fewer the determined dimensions, the more difficult is the work of weight estimation. At the same time the most important weight estimate of all is that which must be made at the very start of the design process when the number of determined dimensions is a minimum. In this chapter, a weight estimate that is made at or near the beginning of the process of design, when few if any of the dimensions of the structural parts to be used are known, will be termed a " preliminary weight estimate."

When an airplane is primarily a revision of an existing design, the preliminary estimation of weights is relatively simple. If the designer knows the weights of the constituent parts of the original design, he has only to estimate the effect of the changes, and this can usually be done rather accurately, especially if the type of construction used is not to be changed. When the type of construction is to be changed, as in designing a metal wing to replace a wood wing, the situation is more like that of a new design, but even here the changes are localized, and the probable percentage error in total weight is small.

The most difficult problem is that confronting the builder of an entirely new design, as he must estimate his weights on the basis of what has been accomplished by other designers on other airplanes, using other types of construction, and other load factors. Under such circumstances, it is hardly to be wondered at that few designers are able to make accurate weight estimates even after they have had considerable experience.

The number of possible variations in size, shape, type of construction, load factors used, and ability of designer is so great that it is not possible to say what any given design ought to weigh, much less what it will weigh. All that can be done is to give some general information as to what different airplanes do weigh, and discuss some of the more

important factors of the problem. It is hoped that this discussion, and the specific data that go with it will aid designers in making their preliminary weight estimates without excessive optimism, so that actual performances will not be disappointing in comparison with predictions.

15 : 2. Classification of Weights — The gross weight of an airplane is divided into two main portions, Useful or Disposable Load and Weight Empty. These main portions are further divided into groups and sub-groups. In general the useful load is composed of the " pay load," including passengers, mail, express matter, baggage, and armament; fuel and oil; special equipment; crew. The main divisions of the weight empty are the structure, power plant, and fixed equipment.

Though there is no question in respect to most of the items constituting the gross weight as to whether they should be counted as part of the useful load or of the weight empty, the exact location of the dividing line is not universally agreed upon. Thus starters are sometimes listed as useful load under special equipment, and sometimes as weight empty under power plant. Instruments are similarly classified variously as useful load under special equipment or as weight empty under fixed equipment.

As the weights of the items of which the classification is unsettled are relatively small, rough comparisons between different designs can be made without knowing where these weights have been allocated. For precise comparisons, however, particularly those between the detailed weights of various airplanes, it is essential that the same items shall be included in each classification for all of the airplanes studied, if the deductions are to be of the greatest possible value. In the formulation of a weight estimate it makes little essential difference how the designer classifies his weights provided that no items are omitted, but it is helpful to him if he uses the same classification as was employed in the data on which he bases his estimate.

In the compilation of data for this chapter the criterion used for distinguishing between useful load and weight empty was that all accessories that would normally be furnished by the manufacturers of commercial airplanes as " standard equipment " were included in the weight empty. Thus starters are so included under power plant. Drift indicators, earth-inductor compasses, and similar instruments not ordinarily carried by airplanes of the types considered were included in the useful load, but the more common instruments as air-speed indicators, altimeters, etc. in the weight empty. All military equipment, other than the standard instruments for each type of airplane, was rated as useful load. Seats are normally considered part of the

weight empty, though the passenger seats of an air-liner could be removed, and would be if the airplane were converted into a bomber in time of war or into a mail or express transport plane. The pilot, though essential to flight, is always considered as part of the useful or disposable load.

15 : 3. Weight of Useful Load — In many specifications, notably those of the Army and Navy, the items to constitute the useful load and the weights to be assumed for them are stated. In others the items to be included as useful load are listed but with incomplete weight data. In the latter case the designer must estimate the weights of the various items, but this can usually be done with a good degree of precision. Sometimes, however, the original specification is so incomplete that the designer must determine what additional items of useful load should be carried as well as estimate their weights. To assist the designer working to an incomplete specification the weights of the chief divisions of useful load of commercial airplanes — personnel, mail, express matter, baggage, fuel, oil, and special equipment — will be discussed in turn.

The weight of a man, who may be either one of the crew or a passenger, varies between rather wide limits and it is common practice to use an assumed figure somewhat greater than the average. For several years the Army and Navy have assumed a man in flying clothes to weigh 180 pounds, exclusive of parachute. Some commercial designers have adopted this practice, but others use 170 pounds in the belief that it is nearer the true average for civilians in ordinary clothing. The Department of Commerce favors 170 pounds.

The weight of any cargo, such as mail, express, or baggage, is usually specified. The question of importance with regard to these items is rather that of the volume of storage space to be allowed for a given weight. First class mail has been found to weigh from 16 to 18 pounds per cubic foot. No general figures can be given for express and baggage as it is too variable in density. The best the designer can do is to provide as much room for such matter as he can without sacrificing performance. If the weight of cargo is not specified, the designer should assume that it is as great as can be carried without reducing the performance below the specification.

If the fuel and oil to be carried are specified in gallons, their weight can easily be computed from the unit weights as given in Table 15 : 16. Sometimes, however, the specification will call for so many hours of flight or miles of cruising range under specified conditions of altitude and engine power. In the general case, the determination of the fuel and oil required to meet a specification of this type involves a perform-

ance computation of the type described by W. S. Diehl in Chapter 12 of "Engineering Aerodynamics" and N.A.C.A. Technical Report No. 234, "Three Methods of Calculating Range and Endurance of Airplanes." Approximate computations can be made from the fuel consumption in pounds per horsepower-hour at rated engine power. For water-cooled engines this figure averages 0.52, and for air-cooled engines 0.55. The deviations from these average figures on account of differences in external conditions and skill of pilots are normally greater than those due to the use of different models of engines. A rule employed by the Air Corps for computing the oil required is that the *volume* of oil should be one-twelfth the *volume* of fuel. Where the application of this rule gives a required oil capacity of 10 gallons or less, the ratio should be one to ten instead of one to twelve.

The equipment considered a part of the useful load, called "special equipment" in this chapter, consists of a number of items such as instruments, other than those furnished as standard with each model, radio, photographic equipment, heating apparatus, and parachutes. Just what a given airplane will be expected to carry in the way of special equipment will depend primarily upon the service in which it is used. Thus a passenger liner would probably carry special lighting equipment and heating apparatus, but no photographic equipment, while on an airplane used in mapping the situation would be reversed. In any event, the special equipment to be carried should be specified, and the weights of the items called for could be learned from their manufacturers. The weights of some items in rather general use are given in Art. 15 : 4.

In estimating the weight of special equipment, the designer should make as complete a list as possible of the items to be carried and their respective weights. He should not take the sum of the weights on this list as his estimate of total special equipment weight but should increase it from 10 to 20 per cent. This increase will provide for a certain amount of added equipment, and also for such items as supporting brackets and clamps, electric wires, and similar items the weights of which depend on the detailed design of the airplane and cannot be controlled by the equipment manufacturer, or readily predetermined by the designer.

The designer need not fear lest by making this and similar allowances too large he may build a structure of excessive weight. If his estimated weight is too great, he will find it impossible to devise a design that will have the aerodynamic performance specified, and will thus be warned to reconsider his weight estimates and see where they can be reduced. If he is able to meet the specified performance with the

excessively estimated weight, advantage can be taken of the conservatism of the estimate by increasing the useful load carried. On the other hand, if the estimated weight is too low, after the airplane is built it may be necessary to reduce the paying portion of the useful load in order to bring the gross weight down to the figure of the preliminary estimate used in the strength computations. Furthermore, designers are optimists by nature, and airplane designs that are built lighter than the figure of the preliminary estimate are few and far between. The writer knows definitely of just one.

15 : 4. Weights of Items of Special Equipment — Most of the equipment and accessories used in an airplane and many of the more standardized structural parts are purchased from their manufacturers by the airplane builder, who has no control over the weights of these items and must take them as they come. If his weight estimate is to be at all correct he must know what the approximate weight of the items he intends to use will be. In order to satisfy the need of the designer for weight data on items of this character, during the summer of 1928 information on weights was requested from the manufacturers of the more important airplane parts and accessories. The data obtained in this manner are listed in the appropriate articles in this chapter. Many of the manufacturers furnished not only weight data but also data on dimensions, capacities, etc. While limitations of space prevent the publication of all of these additional data here, a considerable body of that information which is of special interest to the airplane designer and which lends itself to concise presentation is included in this chapter.

The reader is warned that the weights from different manufacturers are not always strictly comparable although they may appear in the same table. One manufacturer may give an average weight, one competitor a maximum value, and another the normal range of variation. Then too, where the article is not well standardized the manufacturers may differ in the allowances they make for connecting fittings, etc.

As the manufacturers were originally asked for weight data only, the amount of additional data furnished naturally varied greatly. As a result many gaps appear in the tables of data in this chapter. Some of these are due to no information of the kind having been volunteered, and others to the fact that the information was furnished in a somewhat different form. Nearly all of the manufacturers of aircraft equipment have catalogs describing their products in much more detail than is possible in this volume, and airplane manufacturers will find important design data in these catalogs, which will assist them in selecting the sizes and models of the accessories suitable for their own product.

Cameras — Data on cameras were furnished by the Fairchild Aerial Camera Corp., New York City, and the Folmer Graflex Corp., Rochester, N. Y. The more important of these data are listed in Table 15 : 1.

TABLE 15 : 1

DATA ON CAMERAS

Model	Manufacturer	Focal Length of Lens, In.	Size Negative, In.	Size Camera, In.	Weight, Lb.	Operation
K-8	Fairchild	10	$7\times9\frac{1}{2}$	$18\frac{1}{8}\times22\frac{3}{4}\times15\frac{5}{8}$	50[a]	Electric
	Fairchild	12	$7\times9\frac{1}{2}$	$20\frac{1}{8}\times22\frac{3}{4}\times15\frac{5}{8}$	53[a]	Electric
	Fairchild	20	$7\times9\frac{1}{2}$	$30\frac{1}{4}\times22\frac{3}{4}\times15\frac{5}{8}$	62[a]	Electric
A-1	Folmer Graflex	10	4×5	$8\times8\times18$	$13\frac{3}{4}$[b]	Hand
K-5	Folmer Graflex	12	$7\frac{1}{16}\times9\frac{1}{2}$	$19\frac{1}{8}\times19\frac{1}{2}\times16\frac{3}{4}$	37[c]	Hand
	Folmer Graflex	20	$7\frac{1}{16}\times9\frac{1}{2}$	$28\frac{3}{4}\times19\frac{1}{2}\times16\frac{3}{4}$	47[c]	Hand

[a] This weight does not include film. The anti-vibration gimbal mount for the K-8 camera weighs 14 lb. This camera also requires certain accessories, primarily for the production of the electric current and in its operation.

[b] Weight including filled plate magazine.

[c] Weight including 75-ft. film at 3.6 lb.

Parachutes — Weights of parachutes were furnished by Follmer, Clogg & Co., Lancaster, Pa.; Irving Air Chute Co., Buffalo, N. Y.; and the Russell Parachute Co., San Diego, Calif.

The weight of the Follmer, Clogg service type parachute was given as varying between 18 and 19 lb.

The weights of the Irving designs, including harness and attached seat cushions or back-pads, were given as follows:

24-ft. lap type parachute	18 lb.
24-ft. seat type parachute	19.5 lb.
24-ft. back type parachute	17.5 lb.
28-ft. back type parachute	24 lb.
Training outfit	34 lb.

The training outfit is composed of a 28-ft. back type parachute and a 22-ft. parachute carried on the chest. It is designed primarily for military training purposes.

The Russell Co. gave the weight of its parachute with pack and harness as between 18.5 and 19 lb. The addition, often called for, of a small seat cushion and back-pad increases the weight to between 20 and 21 lb. The company recommends that in preliminary estimates the weight of the parachute be assumed as 20 lb.

The size of the Russell parachute folded was stated to be 13 × 15 × 4.5 in.

Radio — The Radio Corporation of America, New York City, furnished data on three types of sending and receiving sets for both telephony and telegraphy. These data are listed in Table 15 : 2. The ranges in this table were given as conservative values which may be considerably exceeded under favorable conditions. The weights given in the table are not final, as the Radio Corp. is redesigning some of the parts with the expectation that their weights will be materially reduced.

TABLE 15 : 2

DATA ON RCA RADIO SETS

Model	Weight, Lb.	Rated Watts in Aerial	Range in Miles	
			Telephony	Telegraphy
ET-3652	$86\frac{1}{2}$	10	25	75
ET-3653	$132\frac{3}{4}$	100	75	300
ET-3654	202	300	200	500

15 : 5. Weight Empty — The estimation of the Weight Empty, including the structure, power plant, and fixed equipment, presents a problem of considerable magnitude. The weight of certain of its elements, notably the engine, instruments, wheels, tires, and similar standard parts, may be determined with considerable accuracy, but the weights of many of their accessories such as tanks, piping, connections, controls, etc., are functions of the particular design and may vary over a considerable range. The weights of the various elements of the airplane structure itself differ greatly depending on the type and arrangement of the members, the materials used and the purpose for which the airplane is designed. Because of the many variables involved it appears to be impossible to give definite weight data which can be applied equally well to all types of airplanes.

If a designer has data on actual weights of parts which he has previously constructed he can generally estimate the weights of similar items for a new design with a high degree of precision. Occasionally he can obtain weights on parts made by other manufacturers and such figures will, of course, assist him in making his own estimate. The correct weights of such parts are, however, generally considered to be "confidential" or "secret" and are not always in agreement with the figures to which the designer may have access, hence too great confidence should not be placed in data obtained from competitors.

Table 15 : 3 was compiled to provide designers with average figures for various items in different types of airplanes. It is the result of a study of weight data on some seventy-five airplanes, about one-third of which were commercial types, the remaining two-thirds being military or naval airplanes. The data on which the table was based were obtained under an agreement that they would be kept confidential, in so far as any specific airplane was involved, and they are believed by the authors to be dependable.

A variation of 10 per cent either way from the tabulated average for any item may be expected since, in any one class, from two to twelve airplanes designed by different men having different facilities and in some cases employing different materials were considered. In a majority of cases the probable variation for any item is 5 per cent.[1]

The upper portion of the table is descriptive in nature and, because of its condensed form, needs some clarification. An externally braced airplane is one in which the wing structure is supported at more than two points along the span, the loads at the outer points of support being transmitted to the fuselage through struts or wires. When the entire wing load is transmitted to the fuselage, or to a cabane structure, by the wing spars themselves, as in the case of a pure cantilever monoplane, the structure is classed as internally braced.

As indicated in the foot-notes, W stands for wooden wing spars, F for fabric covering, and M for metal spars or covering. With one or two exceptions in the large biplane classes, where the wing spars were of steel, the metal used was aluminum alloy. The S used to designate a steel fuselage structure indicates the conventional welded type having either tubular or wire diagonals. Mild carbon, low carbon and chrome molybdenum steels were included, the differences involved being slight in any one type. As used with fuselages the M may denote steel or aluminum alloy, separately or in combination. In the case of the large land monoplanes two aluminum alloy semi-monocoque structures and one welded steel skeleton with aluminum alloy covering were considered. For the observation and bombing airplanes having fabric-covered fuselages the M includes the conventional welded steel tube framework and that in which the forward portion of the fuselage

[1] The authors recognize the lack of completeness of the data on which this table is based, and are very desirous of obtaining additional reliable weight data so that in future editions the table may be corrected to give more representative and up-to-date figures. It is hoped that designers finding the data in this chapter of value to them will contribute data on their own designs. As in the case of the data on which the present edition of the table is based, all such basic data will be kept strictly confidential.

TABLE 15 : 3
Airplane Weight Summary

	Land Monoplanes					Land Biplanes									Sea Biplanes		Amphibians
Gross weight — range	1000	1000	2500	3000	4000	800	1000	2000	2000	2000	3500	4000	6500	10,000	2000	7000	3500
in pounds	2500	2500	3000	4000	5000	1000	2000	2500	2600	3000	5000	7500	10,000	17,000	3500	12,000	5000
Number — crew and passengers	1	2–3	4	5–6	6–7	1–2	2	2–3	2	1	2	2–3	4–5	4–6	2–3	3–4	4–5
Service or commercial	Serv	Com	Com	Com	Com	Com	Com	Com	Serv	Serv	Serv	Serv	Serv	Serv	Com	Serv	Com
Purpose or type	Purs	Gen	Gen	Trans	Trans	Sport	Gen	Gen	Train	Purs	Obsv	Obsv	Trans	Bomb	Trans	Bomb	Trans
External or internal bracing	Ext	Ext	Int	Ext	Both[1]	Ext	Ext	Ext	Ext	Ext	Ext	Ext	Ext	Ext	Ext	Ext	Ext
Wing spars and covering	W.F.	W.F.	W.F.	W.F.	M.M.	W.F.	W.F.	W.F.	W.F.	W.F.	W.F.	M.F.	W.F.	W.M.-F.	W.F.	M.F.	W.F.
Fuselage structure and covering	S.F.	S.F.	S.F.	S.F.	M.M.	S.F.	S.F.	S.F.	S.F.	S.F.	M.F.	M.M.	M.F.	M.F.	S.F.	M.F.	M.-F.M.
Ratio weight empty to gross weight	0.694	0.616	0.615	0.587	0.562	0.623	0.637	0.618	0.699	0.707	0.653	0.603	0.595	0.568	0.641	0.595	0.647
Ratio useful load to weight empty	0.452	0.642	0.626	0.701	0.787	0.629	0.557	0.622	0.433	0.411	0.532	0.672	0.681	0.768	0.565	0.683	0.546
Ratios in per cent of Weight Empty																	
Wing cellule complete	19.2	25.0	25.5	22.1	25.7	26.7	27.8	24.7	26.4	19.1	21.1	26.1	25.7	29.6	23.5	23.7	22.7
Fuselage or hull	14.8	17.7	12.4	17.5	15.3	19.6	16.7	15.0	15.4	11.8	12.0	14.1	14.9	12.7	11.9	13.5[5]	29.5[5]
Nacelles	..	..	..	..	..	..	..	..	..	..	..	..	..	3.4	..	4.1[6]	1.2[6]
Landing gear complete	8.2	8.7	6.5	9.3	6.8	9.3	8.6	8.1	7.4	6.4	6.9	8.0	6.2	8.0	19.5	19.0	8.5
Tail surfaces	2.8	4.2	2.5	3.1	2.9	3.5	3.7	3.6	3.3	2.6	2.5	3.5	2.9	2.7	3.1	2.5	3.7
Power plant (less propeller)	48.8	33.1	40–52[3]	30.0	40.9	34.0	34.2	40.2	38.2	53.5	46.6	34.1	36.8	35.3	32.8	29.6	26.4
Propeller (wood)	3.1	2.8	3.3	3.6[4]	1.7	2.6	2.6	2.7	2.8	2.4	1.7	3.1[4]	1.5	2.6	4.7[4]	3.2[4]	3.7[4]
Fixed equipment	3.7	9.1	3.5	14.6	5.6	4.3	6.4	7.4	6.5	3.6	8.0	11.2	11.5	6.3	4.9	8.3	5.8
Average Weights in Pounds per Unit																	
Wing panel with ailerons — per sq. ft.	0.97	N.D.	..	N.D.	1.56[5]	0.75	1.12	0.92	1.14	1.39	1.25	1.51	1.30	1.41	N.D.	1.48	1.09[6]
Wing cellule complete — per sq. ft.	1.28	1.32	1.72	1.50	2.00[6]	0.91	1.34	1.13	1.46	1.62	1.51	1.78	1.51	1.72	1.50	1.69	1.39[6]
Fuselage or hull — per lin. in.	0.84	0.92	0.93	1.39	1.30	0.49	0.85	0.80	0.99	1.02	1.13	1.43	1.73	1.75	0.92	1.69	1.90[6]
Tail surfaces with bracing — per sq. ft.	0.99	0.99	1.11	1.11	1.08	0.72	0.96	1.04	1.00	1.30	0.97	1.30	0.94	1.06	1.11	1.12	1.04[6]
Radiator, water, and piping — per hp.	0.66	A.C.	0.73	A.C.	0.66	A.C.	A.C.	1.09[7]	1.05[8]	0.63	0.67	0.69	0.72	0.74	A.C.	0.72	A.C.
Gasoline tanks and piping — per gal.	1.61	1.33	1.68	0.77	1.40	1.47	1.68	1.01	1.89	1.84	1.28	0.82	0.95	1.00	1.00	0.93	0.55
Oil tanks and piping — per gal.	2.65	2.62	1.60	2.98	1.48	4.00	2.20	0.00[7]	4.15	2.98	2.43	1.54	2.25	2.24	2.75	2.86	1.75
Average Weight in Pounds for Miscellaneous Items																	
Flooring	6	10–20	10–20	30–40	20–40	N.D.	10	12	16	5	28	25	90	90	10–15	75	25
Firewall	10	10	6–10	12	25	N.D.	3–8	8	6–12	6–12	12–14	10–20	30–50	15–25	6	15–30	6
Surface controls, cables, etc.	10–15	25–50	25–50	30–50	40–60	4–6	20	20	35	20	55	50–75	120	100	25–30	110	55
Instruments and board	10–15	10–15	10–15	15	10–15	6–15	8–15	5–15	16–22	14–17	23–27	25–40	30–40	40–60	12–20	45–60	12–30
Seats and cushions (each)	5–8	10–12	10–12	10–12	10–15	4	4–8	10–12	10–12	12–15	10–12	12–15	10–15	10–12	10–15	10–15	8–10
Baggage or mail compartment	..	10	10	30–120	N.D.	..	10	8	10	..	..	..	N.D.	..	15	..	12
Tail skid	5	5–10	5	10	10–15	3	7	6–8	8–12	6–12	12–18	15–20	20–30	35–45	..	..	25–45
Number of airplanes averaged	3	3	2	2	3	2	5	7	6	10	12	3	3	7	2	3	2

W = Wood. S = Steel. A.C. = Air cooled. F = Fabric. M = Metal. N.D. = No Data. Gen = General. Trans = Transport. Train = Training. Obsv = Observation. Purs = Pursuit. Bomb = Bomber. [1] Av. for 2 int. 1 ext. [2] Air-cooled engine. [3] Water-cooled engine. [4] Metal Propellers. [5] All in Group. [6] One in Group. [7] = OX-5 Only. [8] = Hisso–180 hp.

is welded steel, the rear portion aluminum alloy. With the latter type the dividing line is just aft of the rear cockpit. The metal-covered observation plane fuselages have a welded steel tube framework forward of the rear cockpit, an aluminum alloy monocoque or semi-monocoque structure aft. In all cases the metal covering was aluminum alloy.

15 : 6. Weight of Structure — Of the items given in per cent of weight empty the complete wing cellule includes the wing panels with ailerons, covered, doped and varnished, the interplane struts and flying wires, if any, and the wing fittings. The first two items in the next portion of Table 15 : 3, the weight of the panels with ailerons, covering, etc., and that of the cellule in pounds per square foot, suffice to inform the designer as to the portion of wing weight to be assigned to the panels and the part to be allocated to the struts and wires.

The items covering fuselage, or hull, and the nacelles include these entire structures with covering and cowling but no furnishings. Although it was anticipated that definite figures could be established to indicate the proportion of the weight in the fuselage framework, in the covering, and in cowling, satisfactory averages could not be obtained even with the numerous types used in the classification. It was noted, however, that fuselage structures including covering and cowling were quite constant in weight per inch of length for each of the types, the length of the fuselage being taken as the distance from the front face of the rear propeller flange to the tail post. For two-engined bombers, having no engine in the nose of the fuselage, the length was taken as that from the most forward portion of the fuselage to the tail post. Lack of data on nacelles prevented establishing a similar figure for such structures but as they represent a relatively small part of the total weight in any case a relatively large error in the computation of their weight will have little influence on the value obtained for the preliminary gross weight.

Attention is called to the increase in weight per inch with increase in length of fuselage. This is due, to some extent, to the increased weight involved in obtaining long fuselage structures which have sufficient rigidity to be satisfactory, rigidity very often being of greater importance in the design of large structures than the requirement of strength.

Bare fuselage frameworks for airplanes having air-cooled engines have their centers of gravity located between 30 and 35 per cent of the length of the fuselage behind the front datum point. With water-cooled engine installations they lie between 38 and 45 per cent of the length of the fuselage behind the front face of the rear propeller flange, 41 per cent being a very reasonable figure for a preliminary balance diagram.

The estimation of landing gear weights is not so difficult, since the weight of wheels and tires may be readily obtained from the manufacturers, or, for the more common sizes, from Tables 15 : 5 and 15 : 6. Tail skid average weights are recorded in pounds in Table 15 : 3 so that their variation in weight may be indicated. On a percentage basis tail skids were found to represent about 6 per cent of the total landing gear weight on a majority of the airplanes studied. However, on four or five which were designed with more than the usual regard for maintenance in service, tail skid weights ran from 8 to 12 per cent of the landing gear weight.

For the conventional split-axle type of chassis using rubber cord shock-absorbing devices, the chassis weight including structure, shock-absorbers, and connections is approximately equal to the weight of the wheels and tires, so that doubling the latter figure will give a reasonable value for the landing gear minus tail skid. Where long struts are involved it might be well to assign 60 per cent to the chassis structure, 40 per cent to the wheels and tires. On light airplanes having hydraulic shock-absorbing devices the 60–40 division will give better results but on the heavier types the 50–50 assumption is nearer the average. The use of wheel brakes does not appear to vitiate these assumptions as the increased weight in brake housing, controls, and in the structure is just about equal to that in the wheel itself.

It was expected that good average figures would be obtained for the unit weights of each of the four surfaces of the tail unit, and that a definite percentage could be added to allow for the bracing struts and wires. The range of variation for the individual surfaces was too great to permit averages to be given, but it was found that the average weight of the tail assembly including the braces was remarkably constant when expressed in pounds per square foot of total tail surface area. For a rough but reasonable figure, 75 per cent of the tail surface weight may be assigned to elevator and stabilizer, from 18 to 20 per cent to the vertical surfaces and the rest to bracing.

Especial attention is called to the high weight per square foot required for fast, highly maneuverable airplanes such as pursuit ships. The figures given represent the average for three modern and highly successful designs which were developed as the result of failures in lighter structures.

Shock Absorbing Units — Many airplanes are now being equipped with shock-absorbing units purchased from manufacturers of pneumatic appliances. Data on shock-absorbing struts of this type obtained from the Cleveland Pneumatic Aerol Co., Cleveland, Ohio, and the Gruss Air Spring Co. of America, San Francisco, Calif., are listed in Table

15 : 4. The Cleveland Pneumatic Aerol Co. also makes some special sizes not included in this table.

The weights of the units furnished by the Cleveland Pneumatic Aerol Co. do not include those of the necessary connecting fittings. The Gruss Air Spring Co. did not state how far its weights included such fittings. The dimensions given by the Cleveland Pneumatic Aerol Co. refer to the unit as supplied by it and without the connecting fittings. The Gruss Co. dimensions are between the centers of holes in connecting lugs at the ends of the unit.

TABLE 15 : 4

DATA ON SHOCK ABSORBING STRUTS

Manufacturer	Type	Piston Diam., In.	Static Load Capacity of Pair		Weight of Pair, Lb.	Lengths		
			Maximum, Lb.	Working Range, Lb.		Fully Compressed, In.	Fully Extended, In.	Normal Inflation, In.
Cleveland...	CV16	$1\frac{5}{8}$	1,500	1,000– 1,500	18	$17\frac{7}{8}$	$25\frac{7}{8}$	$22\frac{7}{8}$
Gruss.......		$2\frac{1}{2}$		1,000– 2,000	24	22	30	26
Cleveland...	CV20	2	2,000	1,500– 2,000	21	$17\frac{7}{8}$	$25\frac{7}{8}$	$22\frac{7}{8}$
Gruss.......		3		2,000– 3,000	30	22	30	26
Cleveland...	CV26	$2\frac{5}{8}$	4,000	2,000– 4,000	30	$19\frac{11}{16}$	$27\frac{11}{16}$	$24\frac{11}{16}$
Gruss.......		$3\frac{1}{2}$		3,000– 4,000	36	22	30	26
Cleveland...	CV30	3	6,000	4,000– 6,000	34.5	$19\frac{11}{16}$	$27\frac{11}{16}$	$24\frac{11}{16}$
Cleveland...	CVL[1]	3	6,000	4,000– 6,000	26.5	11	15	14
Gruss.......		4		4,000– 5,000	42	22	30	26
Cleveland...	CV35	$3\frac{1}{2}$	8,000	6,000– 8,000	46	$20\frac{1}{2}$	$28\frac{1}{2}$	$25\frac{1}{2}$
Cleveland...	CV40	4	10,000	8,000–10,000	56	$20\frac{1}{2}$	$28\frac{1}{2}$	$25\frac{1}{2}$
Cleveland...	CV45	$4\frac{1}{2}$	12,000	10,000–12,000	68	$24\frac{1}{2}$	$32\frac{1}{2}$	$29\frac{1}{2}$
Cleveland...	CV47	$4\frac{3}{4}$	15,000	12,000–15,000	72	$22\frac{1}{8}$	$30\frac{1}{8}$	$27\frac{1}{8}$
Cleveland...	CV12[2]	$1\frac{1}{4}$	450[3]	0–450[3]	5.25[3]	$12\frac{1}{8}$	$17\frac{5}{8}$	$16\frac{5}{8}$

[1] For use on amphibians. [2] For tail skids. [3] For single unit instead of pair.

Wheels and Brakes — Data on wheels and brakes were received from the Bendix Brake Co., South Bend, Ind., and the Kelsey-Hayes Wheel Corp., Jackson, Mich., on the wheels and brakes of their own manufacture; and from the Johnson Airplane and Supply Co., Dayton, Ohio, on wheels made by the Dayton Wire Wheel Co. The weight data received from these companies are listed in Table 15 : 5.

On account of the recent activities of the S. A. E. whose committee on the subject has adopted a tentative standard for airplane wheels, only those sizes are shown in the table that are listed in the tentative S. A. E. standard. The strengths and axle dimensions given in the table are those of the tentative S. A. E. standard rather than those

TABLE 15:5

DATA ON WHEELS

Size	Static Strength, Lb.		Working Load, Lb.	Inflation Pressure, Lb.	Hub		Oversize Tire	Weight in Pounds				
								Plain Wheel			Wheel with Brake	
	Radial	Side			Length, In.	Diameter, In.		Bendix	Dayton	Kelsey-Hayes	Bendix	Kelsey-Hayes
14×3	5,000	1,800	400	50	4.00	0.745 1.245	none	[a]	3	3.1	[a]	....
18×3	6,000	2,000	475	50	5.00	1.245	20×4	[a]	6	5.1	[a]	....
24×3	7,500	2,500	550	50	6.00	1.495	26×4	....	8.8	7.4	15	....
24×4	6,000		800	50	5.00	1.500	26×5	....	....	7.9	....	....
28×4	10,000	2,750	1000	50	7.25 6.00	1.688	30×5	....	12.5	11.25	17	18.5
30×5	11,000	3,300	1600	50	7.25	2.188	32×6	13	....	12.75	23	24
32×6	13,500	4,050	2200	55	7.25	2.188	36×8	18	19	18.75	29	....
36×8	20,000	6,000	4000	60	7.25	2.688	40×10	20	26	28.5	31	....
44×10	40,000	12,000	6000	65	10.00	3.188	none	40	67	....	65	....
54×12	50,000	15,000	9000	70	12.00	3.937	none	....	....	....	100	....

[a] Under Design.

pertaining to the wheels of the separate manufacturers, which in several cases are somewhat different.

The working load values in the table are obtained by dividing the radial strength by an assumed load factor. In design it is not safe to rely on these working load figures, but the radial strength of the wheel should be divided by the load factor in landing required for the design in question to determine the working load in any specific case.

The weights from the Bendix Co. were given as approximate, and those from Kelsey-Hayes as maximum values. In the other case the character of the weight figures was not stated.

Tires — Data on tires were obtained from the B. F. Goodrich Rubber Co. and the Goodyear Tire and Rubber Co., both of Akron, Ohio. The weights and rated load capacities of those tires listed in the tentative S. A. E. standard for wheels and tires are given in Table 15 : 6.

As it was the opinion of the S. A. E. committee drawing up the tentative standard that oversize tires should be used only to increase contact area with the ground and not to carry extra load, the rated strengths for the tires used only as oversizes have been omitted.

In addition to the sizes of tires listed in the table, both manufacturers continue to make tires to fit other rim sizes and also tires of extra strength for special conditions.

TABLE 15 : 6

DATA ON TIRES

Size	Air Pressure, Lb. per sq. in.	Loads in Pounds		Weight in Pounds of Casing, Tube and Flap [c]			
				Smooth Tread		Non-skid Tread	
		Rated	Collapsing[a]	Goodrich	Goodyear	Goodrich	Goodyear
14×3	50	400	.....	3.9	.....	4.0	2.7
18×3	50	475	.....	4.9	.....	.....	.....
20×4	50	b	b	7.1	.....	8.5	7.7
24×3	50	550	.....	6.8	.....	.....	.....
26×4	50	b	b	.....	.....	10.8	10.5
28×4	50	1000	3,950	9.9	11.1	12.0	11.1
30×5	50	1600	5,400	13.1	14.5	15.9	14.5
32×6	55	2200	8,200	17.4	17.3	19.0	17.3
36×8	60	4000	14,300	30.9	32.6	33.5	32.6
40×10	60	b	b	.....	.....	54.7	54.7
44×10	65	6000	22,600	58.9	65.2	.....	.....
54×12	70	9000	39,000	106.4	99.0	.....	.....

[a] Collapsing load is that causing 100 per cent deflection.

[b] Used only as oversize.

[c] Goodrich figures include no flap.

15 : 7. Weight of Power Plant — The weight of the power plant should be first estimated by adding the estimates of the weights of its constituent parts, such as the engine, propeller, fuel system, oil system, starting system, cooling system, engine controls, and exhaust manifolds. These unit estimates may be made from the data given below, some of which were obtained from manufacturers of engines and accessories and some from a study of existing designs. A rough check of the estimated total weight of the power plant can be obtained by comparison with the values listed in Table 15 : 3 for the ratios of power plant minus propeller and of propeller alone to the weight empty. The reason for having separate figures for the propeller and the rest of the power plant was the relatively large difference in the weights of wood and metal propellers for airplanes of the same general type. Additional checks obtained from the study of existing power plants are as follows:

1. With water-cooled engines the engine weight, dry, normally accounts for from 55 to 65 per cent of the complete power plant, with the complete cooling system accounting for about 22 per cent additional.

2. Air-cooled engines normally constitute from 75 to 85 per cent of the power plant weight, 80 per cent being a representative value.

3. Propellers, fuel and oil systems, engine controls, and exhaust manifolds together normally weigh between 0.5 and 0.7 lb. per engine horsepower, but this figure should be larger for airplanes carrying more than four or five hours' supply of fuel or equipped with tanks of relatively heavy construction.

Before applying the checks just suggested the designer should consider whether his design is sufficiently conventional so that they will apply, and if not, in which direction the deviation should appear. By an intelligent use of the checks, he should then be able to correct his original estimates to obtain one that will agree satisfactorily with the power plant as constructed. A very close agreement, however, cannot be expected by the inexperienced designer as the precise allowances to be made for many of the factors involved can be determined only by judgment resulting from experience. It is believed, however, that for most designs, if the values suggested in the text are adhered to conservatively, the result will be a weight estimate that can be met, and in many cases improved upon, by careful design of details.

Engines — The largest and most important element of the power plant is the engine. Table 15 : 7 copied from the December 1, 1928, number of *Aviation* gives the weights and other data on the engines commercially available in this country at that time. As this table is

corrected monthly the practicing designer should take his figures from the latest issue.

Propellers — The weight of the propeller depends on a number of factors, the most important being the material used, the power of the engine, and the diameter of the propeller. The last named quantity depends largely on the engine power, the speed of the propeller in r.p.m., and the velocity of the airplane in miles per hour. A good rough estimate of weight for normal designs of wood propellers can be made from the horsepower alone, using the formula

$$\text{Weight of propeller in pounds} = (0.084 \times \text{hp. of engine}) + 23$$

The weights obtained from this formula will not be exact as it does not take into account the diameter of the propeller or the species of wood used. Since the diameter increases as the design speed of the airplane decreases, somewhat heavier weights should be estimated for slow airplanes, while somewhat smaller values could be used for fast ones. The normal range of variation from the values given by the formulas is about ± 23 per cent. For the propeller used with a 90-hp. engine, the formula gives a weight of 31 lb., but the weights of several propellers used on commercial airplanes with the OX-5 engine weigh from 35 to 37 lb., the error of the formula being in this case from 13 to 19 per cent. For a 500-hp. engine, the range of probable error of the formula is ± 15 lb. This formula should be used only for wood propellers of normal design directly connected to the crankshafts of conventional engines. For propellers used with geared engines or engines of the cam type, much heavier weights should be estimated on account of the lower propeller speeds and resulting greater diameters and weights.

Aluminum Alloy Propellers — Data on aluminum alloy propellers were obtained from the Curtiss Aeroplane and Motors Co., Garden City, Long Island, N. Y., and the Standard Steel Propeller Co., Pittsburgh, Pa. They are listed in Table 15 : 8. These weights include that of the hub and can be added to the weights of engines given in Table 15 : 7 to obtain the weight of the complete engine-propeller unit.

The Standard Steel propellers, and that of the Curtiss company used with the 600-hp. GV-1550 engine, are of the adjustable pitch type. The other Curtiss designs are of the fixed pitch type.

TABLE 15:7

DATA ON ENGINES AVAILABLE FOR COMMERCIAL USE

Manufacturer or Agent	Engine	Cooling System	Cylinder Arrangement	No. of Cylinders	Propeller Drive	Rated Hp. and R.p.m. at Sea Level	Maximum Hp. and R.p.m. at Sea Level	Weight Dry Without Hub or Starter	Weight per Rated Hp.	Bore and Stroke (In.)	Diameter of Mounting Ring or Distance Between Bearers (In.)	Height or Overall Diameter (In.)	Length Without Starter (In.)	Guaranteed Fuel Consumption at Rated Hp. Lb./Hp./Hr.	Oil Consumption Lb./Hp./Hr.	Type of Ignition	Make Starter	Are Exhaust Manifolds Provided
Aircraft Engine Co.	Comet	A	Rad	7	D	130–1825	....	....	...	$4\frac{1}{2}\times5\frac{1}{2}$	......	......	......	......	......	2M	Bs	....
Allison Engineering Co.	V1410	A	IV-45°	12	D	410–1800	430–1900	1000	2.44	$4\frac{5}{8}\times7$	17	$46\frac{3}{4}$	$78\frac{1}{2}$	.55	.03	2B	Ec	....
Allison Engineering Co.	VG1410	A	IV-45°	12	G[1]	410–1800	430–1900	1125	2.62	$4\frac{5}{8}\times7$	17	......	$85\frac{1}{8}$	.55	.03	2B	Ec	....
Allison Engineering Co.	V1650	W	IV-45°	12	D	420–1700	450–1800	885	2.10	5×7	17	$46\frac{7}{16}$	$74\frac{1}{4}$	.53	.025	2B	Ec	....
Axelson Machine Co.	Floco	A	Rad	7	D	115–1800	....	430	2.87	$4\frac{1}{2}\times5\frac{1}{2}$	......	$45\frac{1}{2}$	35	......	......	M	......	....
Brownback Motor Laboratories, Inc.*	Anzani 3-35	A	Rad	3	D	35–1700	45–1800	115	3.29	$3\frac{37}{64}\times4\frac{23}{32}$	9.5	$24\frac{13}{16}$	$21\frac{13}{32}$	.53	.025	2M	......	N
Brownback Motor Laboratories, Inc.*	Anzani 6-80	A	Rad	6	D	80–1500	89–1600	215	2.69	$4\frac{1}{8}\times4\frac{15}{16}$	10.6	$35\frac{13}{16}$	29	.53	.027	2M	......	N
Brownback Motor Laboratories, Inc.*	Anzani 10-120	A	Rad	10	D	120–1500	131–1600	320	2.67	$4\frac{1}{8}\times5\frac{1}{2}$	11.0	$43\frac{5}{16}$	$34\frac{11}{32}$	.60	.031	2M	......	N
Curtiss Aeroplane & Motor Co.,	OX-5	W	V-90°	8	D	90–1400	....	375	4.17	4×5	$12\frac{7}{8}$	......	......	.55	.030	1M	......	N
Curtiss Aeroplane & Motor Co.,	OXX6	W	V-90°	8	D	110–1900	....	390	3.55	$4\frac{1}{4}\times5$	$12\frac{7}{8}$	......	......	.55	.030	2M	......	N
Curtiss Aeroplane & Motor Co.,	C-6A	W	L	6	D	160–1750	165–1750	420	2.62	$4\frac{1}{2}\times6$	$15\frac{3}{4}$	39	$57\frac{15}{32}$	.52	.015	2M	U	N
Curtiss Aeroplane & Motor Co.,	D12	W	V-60°	12	D	435–2300	460–2500	680	1.47	$4\frac{1}{2}\times6$	$15\frac{3}{4}$	......	$56\frac{3}{4}$	.52	.015	2M	Ec	N
Curtiss Aeroplane & Motor Co.,	Conqueror	W	V-60°	12	D	600–2400	635–2450	760	1.26	$5\frac{1}{8}\times6\frac{1}{4}$	$15\frac{3}{4}$	......	$56\frac{3}{4}$	.52	.015	1M	Ec	N
Curtiss Aeroplane & Motor Co., Inc.	Geared Conqueror	W	V-60°	12	G[2]	600–2400	625–2450	850	1.42	$5\frac{1}{8}\times6\frac{1}{4}$	$15\frac{3}{4}$	......	$56\frac{5}{32}$	.52	.015	1M	Ec	N
Curtiss Aeroplane & Motor Co.	Chieftain	A	H	12	D	600–2200	615–2200	900	1.50	$5\frac{5}{8}\times5\frac{1}{2}$	$23\frac{3}{8}$	$45\frac{1}{4}$	$51\frac{11}{16}$	.60	.025	DM	E	N
Curtiss Aeroplane & Motor Co.	Challenger	A	Rad S	6	D	170–1800	180–1800	420	2.47	$5\frac{1}{4}\times4\frac{7}{8}$	19	$41\frac{3}{4}$	$35\frac{5}{32}$	.55	.020	2M	E	N
Dayton Aeroplane Engine Co.	Bear	A	L	4	D	76–1425	....	375	3.41	$4\frac{1}{2}\times7$	16	39	$47\frac{5}{8}$	.505	.0175	2M	N	N
Fairchild Caminez Eng. Corp.	Caminez 447-C	A	Rad	4	D	120–960	140–1050	350	2.50	$5\frac{5}{8}\times4\frac{1}{2}$	$21\frac{1}{2}$	36	35	.55	.035	2M	Ec	....

Hallet Mfg. Co.	H-526	A	Rad	7	D	130–1800	......	425	3.26	$4\frac{3}{8}\times 5$	$18\frac{1}{2}$	46	37	......	......	2M	......	N
Irwin Aircraft Co.	Irwin 79	A	Rad	4	D	20–1700	25–2200	66	3.00	$2\frac{7}{8}\times 2\frac{3}{4}$	$4\frac{7}{8}$	23	11	.45	.031	1M	......	N
Kinner Airplane & Motor Corp.	K-5	A	Rad	5	D	90–1810	......	268	2.30	4.25×525	14	42	......	......	......	2M	H	Y
Menasco Motors Co.	B-2 Salmson	A	Rad	9	D	260–1500	278–1760	542	2.08	4.92×6.69	......	$49\frac{1}{8}$	$38\frac{1}{2}$	.42	.025	2M	......	...
O. E. Szekely Corp.	SR-3	A	Rad	3	D	40–1800	45 6–1900	148	3.7	$4\frac{1}{8}\times 4\frac{3}{4}$	$9\frac{7}{8}$	......	$23\frac{3}{4}$	.57	.030	2M	N	N
Packard Motor Car Co.	IA-2775	W	X-60°	24	D	1200–2600	1300–2700	1500	1.15	$5\frac{3}{8}\times 5$	$14\frac{1}{4}$	$45\frac{1}{32}$	$78\frac{1}{16}$	.52	.03	4B	O	...
Packard Motor Car Co.	3A-1500	W	V-60°	12	D	525–2100	......	760	1.26	$5\frac{3}{8}\times 5\frac{1}{2}$	$15\frac{3}{4}$	$38\frac{15}{64}$	$62\frac{11}{32}$	.52	.015	2M	Ae	N
Packard Motor Car Co.	3A-1500	W	V-60°	12	G	600–2500	650–2700	880	1.46	$5\frac{3}{8}\times 5\frac{1}{2}$	$15\frac{3}{4}$	$38\frac{15}{64}$	65	.52	.025	2B	Ae	N
Packard Motor Car Co.	3A-1500	W	V-60°	12	D	600–2500	......	780	1.30	$5\frac{3}{8}\times 5\frac{1}{2}$	$16\frac{1}{4}$	$38\frac{5}{8}$	$62\frac{11}{32}$	.52	.025	2M	Ae	N
Packard Motor Car Co.	3A-2500	W	V-60°	12	D	800–2000	835–2100	1160	1.45	$6\frac{3}{8}\times 6\frac{1}{2}$	$18\frac{1}{4}$	$36\frac{1}{8}$	$69\frac{3}{4}$	.52	.015	2M	Ae	N
Packard Motor Car Co.	3A-2500	W	V-60°	12	G	800–2000	835–2100	1380	1.72	$6\frac{3}{8}\times 6\frac{1}{2}$	$18\frac{1}{4}$	$36\frac{1}{8}$	$76\frac{1}{2}$	.52	.015	2B	Ae	N
Pratt & Whitney Aircraft Co.	Wasp	A	Rad	9	D	425–1900	......	670	1.67	$5\times 5\frac{3}{4}$	$23\frac{3}{8}$	......	$43\frac{3}{8}$	.55	.035	2M	......	N
Pratt & Whitney Aircraft Co.	Hornet	A	Rad	9	D	525–1900	......	760	1.52	$6\frac{1}{8}\times 6\frac{3}{8}$	$23\frac{3}{8}$	......	$44\frac{5}{8}$	.55	.035	2M	......	N
Pratt & Whitney Aircraft Co.	Geared Hornet	A	Rad	9	G^2	500–1900	......	835	....	$6\frac{1}{8}\times 6\frac{3}{8}$	$23\frac{3}{8}$	......	$48\frac{3}{4}$	.55	.035	2M	......	N
Quick Motors Co.	Quick-Radial	A	Rad	9	D	125–1450	......	325	2.6	4.13×5.51	$19\frac{3}{16}$	36	36	......	......	1M	......	...
Siemens & Halske A.G.*	Siemens Sh 13	A	Rad	5	D	80–1750	......	246	3.01	4.13×4.72	14	$39\frac{5}{8}$	34	.52	.022	2M	O	Y
Siemens & Halske A.G.*	Siemens Sh 14	A	Rad	7	D	115–1750	......	308	2.68	4.13×4.72	14	$39\frac{5}{8}$	32	.52	.022	2M	O	Y
Siemens & Halske A.G.*	Siemens Sh 12	A	Rad	9	D	122–1575	......	382	3.05	3.94×4.72	14.6	$40\frac{1}{2}$	32	.52	.022	2M	O	Y
Spartan Aircraft Co.	Walter 5	A	Rad	5	D	70–1600	......	225	3.21	4.13×4.72	11.35	37	27.5	.50	.028	2M	......	Y
Spartan Aircraft Co.	Walter 7	A	Rad	7	D	95–1600	......	280	2.95	4.13×4.72	12.6	37	28.5	.49	.026	2M	......	Y
Spartan Aircraft Co.	Walter 9	A	Rad	9	D	120–1600	......	352	2.59	4.13×4.72	14.56	39	33.5	.49	.026	2M	......	Y
Tips & Smith, Inc.†	Super-Rhone	A	Rad	9	D	125–1450	133–1650	315	2.52	$4\frac{1}{8}\times 5\frac{1}{2}$	11	......	39	.31	.036	1M	......	...
Velie Motors Corp.	Velie M5	A	Rad	5	D	55–1815	......	210	4.66	$4\frac{1}{8}\times 3\frac{3}{4}$	$12\frac{3}{4}$	32	27	.55	.025	2M	Ec	Y
Warner Aircraft Corp.	Scarab	A	Rad	7	D	110–1850	......	270	2.45	$4\frac{1}{4}\times 4\frac{1}{4}$	17	$35\frac{1}{2}$	$27\frac{1}{2}$	.50	.010	2M	......	N
Wright Aeronautical Corp.	Whirlwind J5	A	Rad	9	D	225–1800	......	500	2.50	$4\frac{1}{2}\times 5\frac{1}{2}$	$19\frac{1}{4}$	45	$33\frac{3}{4}$	.53	.035	2M	......	N
Wright Aeronautical Corp.	Cyclone R1750	A	Rad	9	D	525–1900	560–2000	760	1.52	$6\times 6\frac{7}{8}$	$23\frac{3}{8}$	$53\frac{15}{16}$	$40\frac{5}{8}$	.55	.035	2M	......	N

* — Engines of foreign manufacture
† — rebuilt
A — Air cooled
Ae — Aeromarine
B — Battery ignition
Bs — Scintilla booster magneto

D — Direct
DM — Dual Magneto
Ec — Eclipse
E — Ensign
G — Geared
G^1 — 5:3 epicyclic gear

G^2 — 2:1 reduction gear
H — Hexagonal
I — Inverted
L — Cylinders in line
M — Magneto
N — No

O — Own
Rad — Radial
Rad S — Staggered Radial
V — Vee
W — Water cooled
Y — Yes

The values in this table were obtained from *Aviation* for Dec. 1, 1928. They are believed to be dependable but are not guaranteed as to accuracy.

TABLE 15 : 8

DATA ON REPRESENTATIVE ALUMINUM ALLOY PROPELLERS

Engine	Rated Hp.	Rated Normal Propeller R.p.m.	Airplane High speed, M.p.h.	Propeller Diameter		Propeller Weight, Lb.	Manufacturer
				Ft.	In.		
Curtiss OX-5	90	1400	100	8	6	52	Standard
Curtiss OX-5	100	...	70	8	6	53	Curtiss
C-6	160	1750	116	9	0	70	Curtiss
Wright E-4	200	...	130	8	9	88.5	Curtiss
Wright J-5	220	...	130	9	0	90	Curtiss
Wright Whirlwind	230	1900	130	9	0	82	Standard
Wright Whirlwind	230	1900	110	9	6	85	Standard
Liberty 12	410	1750	147	10	0	95	Curtiss
"	410	1750	140	9	2	85	Standard
"	410	1750	160	8	10	82	Standard
"	410	1750	180	8	6	79	Standard
Pratt and Whitney Wasp	430	1950	160	9	9	90	Standard
"	430	1950	157	9	3	77*	Curtiss
"	430	1950	100	11	0	110	Standard
"	430	1950	120	10	6	103	Standard
"	430	1950	140	10	0	96	Standard
"	430	1950	140	10	0	92	Standard
"	430	1950	120	10	3	96	Standard
Curtiss D-12	435	2300	143	9	0	95	Curtiss
Pratt and Whitney Hornet	535	1900	120	11	0	125	Standard
"	535	1900	120	10	6	118	Standard
Packard 1500	540	2100	150	9	6	90	Standard
Packard 1500	600	2500	180	8	6	80	Standard
Packard 1500 geared	600	1250	120	14	0	260	Standard
" "	600	1250	110	12	10	300**	Standard
Curtiss V-1550	600	...	164	8	9	95	Curtiss
Curtiss GV-1550	600	...	130	14	0	208	Curtiss
Packard 2500 geared 2 : 1	800	1050	130	16	0	380	Standard

* Splined directly on engine without use of steel sleeve.

** Three blades.

Fuel System — The fuel system is composed of the fuel tanks and piping, the former being much the more important. The weight of fuel tanks per gallon capacity varies with shape, size, and material, to such an extent that no definite formula for this quantity can be

given. The weight of piping varies largely with the length of the fuel lines as well as their size and material. For preliminary estimates it is reasonable to combine the two items and estimate a weight per gallon capacity for the entire fuel system. On over 50 military airplanes this varied from 0.6 to 2.6 lb. per gallon.

For airplanes carrying 60 gal. or less of fuel, it is advisable to assume the fuel system at 2.0 lb. per gallon. From 60 to 130 gal., the figure would be 1.5 lb. per gallon; from 130 to 500 gal., 1.0 lb. per gallon; and above 500 gal., 0.7 lb. per gallon. These figures are conservative and may be reduced somewhat for aluminum tanks, but should be increased if leak-proof or crash-proof tanks are used. About 80 per cent of the total weight of the fuel system should be assigned to the tanks and the remainder to the piping.

Representative values for fuel systems are given in Table 15 : 3 for various types of airplanes.

Oil System — The oil system consists of the oil tank and piping and the oil radiator, if any. Owing to the volume of oil carried being much smaller than that of fuel, the weight of the oil system per gallon is usually about twice that of the fuel system. The difference in per cent may be large, but the total weight involved is small so twice the unit weight assumed for the fuel system may be used for the oil system of the same airplane. If an oil radiator is used, the unit weight of the oil system should be doubled, as the weight of the radiator is usually about equal to that of the oil tank and piping. In dividing the weight of the oil system between the tank and the piping, about 70 per cent should be allotted to the tank.

Representative weights of oil systems are given in Table 15 : 3 for various types of airplanes. The Curtiss OX-5 engine, which carries its oil in the engine and requires no external oil system, was used in all seven of the 2–3 place commercial types for which the weight of the oil system is given as 0.00 lb. per gallon.

Starters — The data on starters listed in Table 15 : 9 were furnished by the Aeromarine Starter Co., Keyport, N. J., and the Eclipse Machine Co., East Orange, N. J.

The Aeromarine starter weights do not include the weight of the hand crank, which is 3 lb. for all models.

The Eclipse weights do not include the following units:

Solenoid switch	3 lb.
Push and pull starting switch	$2\frac{1}{2}$ oz.
Crank handle	$2\frac{3}{4}$ lb.
Crank extension tubing	$1\frac{3}{4}$ lb.

The Eclipse company also makes hand-turning gear for cranking the engine directly by hand power, these gears generally being used in connection with a booster magneto. Type 1H-6 without the booster magneto weighs 17.5 lb. Type 1HB-6 with the booster magneto weighs 28 lb. This size is designed for engines up to 900-cu. in. piston displacement.

The Aeromarine company also makes a prompt-mesh (non-inertia) electric starter weighing 24 lb. complete, suitable for use with engines up to the size of the Liberty.

TABLE 15 : 9

WEIGHTS OF AIRPLANE STARTERS

<table>
<tr><th rowspan="3">Manu-facturer</th><th rowspan="3">Series or Type</th><th rowspan="3">Maximum Engine Displace-ment, Cu. in.</th><th colspan="4">Weight in Pounds</th><th rowspan="3">Remarks</th></tr>
<tr><th colspan="2">Hand Operated</th><th colspan="2">Electric Operated</th></tr>
<tr><th>Land Ser.</th><th>Marine Ser.</th><th>Land Ser.</th><th>Marine Ser.</th></tr>
<tr><td>Aero-marine</td><td>C</td><td>1350</td><td colspan="2">17</td><td></td><td></td><td></td></tr>
<tr><td>Eclipse</td><td>6</td><td>1350</td><td>17.5</td><td>19.5</td><td>24.5</td><td>27.5</td><td>Primarily for radial engines</td></tr>
<tr><td>Aero-marine</td><td>D</td><td>2500</td><td colspan="2">24</td><td colspan="2">30</td><td></td></tr>
<tr><td>Eclipse</td><td>11</td><td>2500</td><td>19.5</td><td>21.5</td><td>28.5</td><td>30.5</td><td>Primarily for radial engines</td></tr>
<tr><td>Eclipse</td><td>7</td><td>2500</td><td colspan="2">27</td><td colspan="2">35</td><td>Primarily for V-type engines</td></tr>
</table>

Cooling System — With air-cooled engines there is no cooling-system weight to be considered as it is included in the weight of the engine. With water-cooled engines, however, the designer must estimate the weights of the radiator, expansion tank, water piping, and the water in those units and in the engine. Table 15 : 10 compiled from Air Corps data gives the weight of water carried in the more important American water-cooled engines and estimates of the total weights of radiator, expansion tank, and included water. The rated horsepowers of the engines are also given as an aid in estimating the same quantities for other engines. The weight of the piping and included water is not covered by the figures in the table, but may be estimated at from 10 to 25 lb. depending on the size of the engine and the location of the radiator.

Study of the detailed weights of a number of Air Corps airplanes showed that in the great majority of cases the total weight of the cooling

TABLE 15 : 10

ESTIMATED WEIGHTS OF COOLING SYSTEMS FOR VARIOUS ENGINES

Engine	Horsepower	Weights in Pounds	
		Engine Water	Radiator, Tanks, and Water
Curtiss OX-5	90	17	75
Curtiss D-12	430	44	160
Curtiss V-1400	500	47	195
Liberty 12	420	45	190
Packard 1A-1500	510	24	215–245
Packard 1A-2500	800	40	380
Wright I and E-2	190	44	110

system divided by the rated engine horsepower ranged between 0.6 and 0.8 lb. per horsepower. As might be expected, however, from the figures given in Table 15 : 10 the cooling system weights for airplanes with the OX-5 and Wright E engines were considerably higher, being in some cases slightly over 1.0 lb. per horsepower. Some specific figures for various types are given in Table 15 : 3. For preliminary estimates it is advisable to assume a weight of 0.7 lb. per horsepower for most designs. For the two engines just mentioned 1.0 would be a better figure to use. As the weight of cooling water increases with decrease in speed, the unit weights just mentioned might be slightly decreased for fast airplanes, but should be increased for slow ones.

Additional data on radiator weights, supplied by the Winchester Repeating Arms Co. of New Haven, Conn., are given in Table 15 : 11. These data apply to the hexagonal-cell type radiator manufactured by that company.

TABLE 15 : 11

RADIATOR DATA FROM WINCHESTER REPEATING ARMS CO.

Length of Tube, In.	"F" In.	Per Square Foot Frontal Area				
		Cooling Surface, Sq. ft.	Water Space, Cu. in.	Water Weight, Lb.	Tube Weight, Lb.	Free Air Area, Sq. ft.
4	0.313	39.3	175	6.3	13.5	0.606
5	0.313	48.43	223	8.0	19.8	0.606
6	0.308	60.07	248	8.9	23.6	0.626
7	0.308	69.86	293	10.5	27.6	0.626
9	0.308	89.43	384	13.8	35.6	0.626

"F" is the diameter of the inscribed circle of the hexagonal air passages.

The data in Table 15 : 11 apply only to the core of the radiator. In estimating the total weight an allowance of 20 per cent of the weight of the dry core should be added to allow for the shell, mounting brackets, etc. A similar allowance of 20 per cent of the weight of the water in the core should be made to cover the water in the tanks. In determining the size of radiator to be used, an allowance of $\frac{3}{4}$ sq. ft. of radiating surface per horsepower of the engine may be used. This is not an absolute figure, however, as the efficiency of the surface increases with increase of speed of the airplane. For slow designs, therefore, approximately 10 per cent more surface should be provided.

The amount of radiating surface needed cannot be stated definitely as the efficiency of the surface depends partly on the speed of the airplane, partly on the temperature of the air, and very largely on the location of the radiator. Much trouble has been experienced in the past with the cooling systems of new designs, and it is advisable to err by providing too large a radiator rather than one that is too small. In some designs the radiators are provided with shutters, and these add a few pounds to the weight.

Engine Controls — The engine controls constitute only a very small part of the weight of the power plant, seldom representing as much as one per cent of that quantity. On single-engined airplanes with the engine fairly close to the pilot, they range in weight from 3 to 10 lb., usually weighing about 5 lb. On multi-motored airplanes where the engine is relatively remote from the pilot, the controls will weigh from 20 to 36 lb.

Exhaust Manifolds — The exhaust manifolds are another minor item of weight. For engines under 250 hp. they average 8 to 12 lb. in weight. For the larger engines the weights run from about 16 to 25 lb., most designs weighing very close to 20 lb. The weight of exhaust stacks depends very much on their length; and as the above figures were taken from military designs they should be considerably increased for commercial designs in which the exhaust is carried back behind the passengers' cabin. Any silencing device would also add to the weight of this unit, but this added weight might be charged up to Fixed Equipment instead of Power Plant.

15 : 8. Weight of Fixed Equipment — The following items may be included under fixed equipment: flooring, firewall, surface controls with cables and rods, instruments and instrument board, seats and cushions, upholstery, baggage or mail compartments, lighting and heating equipment, and miscellaneous fixed items which would be included as "standard equipment" by the manufacturer. As is apparent, the percentage averages shown in Table 15 : 3 vary considerably, and little dependence

should be placed on them. The average weights for specific items, however, were reasonably constant for each of the types and so could be tabulated, the range of weight in pounds being indicated wherever average weights would not be representative. Data on various specific items of fixed equipment usually purchased by the airplane manufacturers are given below.

Generators — Data on generators were furnished by the Eclipse Machine Co., East Orange, N. J., and the Leece-Neville Co., Cleveland, Ohio. These data are listed in Table 15 : 12.

Instruments — Weights of instruments were obtained from the Consolidated Instrument Company of America, New York City; Elgin National Watch Co., Chicago, Ill.; Johnson Airplane and Supply Co., Dayton, Ohio; Pioneer Instrument Co., Brooklyn, N. Y.; and Taylor Instrument Companies, Rochester, N. Y.

In most cases the manufacturers gave the weights of these instruments in ounces, but as such precision is unnecessary in preliminary weight estimates, the weights listed in Table 15 : 13 are given only to the nearest pound or half pound. Also, as the weights of the similar instruments made by different manufacturers differ very little, only a single figure or range of figures is given for each item. In some cases where a range of figures is given for an item it is due to the differing weights of the products of different manufacturers, in others it is due to the inclusion of two or more models of the same class of instrument under the same item.

TABLE 15 : 13

WEIGHTS OF INSTRUMENTS

Instruments	Weight, Lb.	Source of Information
Air Speed Indicator	1–2	Consolidated, Johnson, Pioneer
Altimeter	1	Consolidated, Pioneer, Taylor
Clock	1	Pioneer
Compass, Magnetic	$1\frac{1}{2}$–4	Consolidated, Elgin, Johnson, Pioneer
Compass, Induction	16	Pioneer
Flex. Shafting, per ft.	$\frac{1}{2}$	Elgin
Flight Indicator	2–3	Pioneer
Fuel Level Indicator	1–2	Pioneer
Lateral Inclinometer	$\frac{1}{2}$	Consolidated, Pioneer, Taylor
Oil Pressure Gage	$\frac{1}{2}$	Consolidated, Pioneer
Pitot Tube	$\frac{1}{2}$	Pioneer
Rate of Climb Indicator	1	Pioneer
Speed and Drift Indicator	2	Pioneer
Tachometer	1	Consolidated, Elgin, Johnson, Pioneer
Tachometer Adapter	$\frac{1}{2}$	Johnson, Pioneer
Temperature Gage	1	Consolidated, Pioneer, Taylor
Turn Indicator	1–2	Pioneer
Venturi Tube	$\frac{1}{2}$	Pioneer

TABLE 15 : 12

DATA ON GENERATORS

Type	Manufacturer	Min. Gener. Speed to Carry Rated Load, R.p.m.	Normal Generator Speed, R.p.m.	Rated Voltage, Volts	Rated Line Currents, Amperes	Outside Diameter, In.	Direction of Rotation at Drive End[b]	Weight Generator Only, Lb.	Weight of Control Box, Lb.[d]	Total Weight, Lb.
CG-1	Leece-Neville	2250	2750	12–15	15	4	C.C.	16	4	20
G-1	Leece-Neville	2250	2750	12–15	15	4	C.C.	16	3	19
G-2	Leece-Neville	2250	2750	12–15	15	4	C	16	3	19
G	Eclipse	2250	....	15	15	4.5	E	15	2	17
F-1[a]	Leece-Neville	2000	2250	12–15	15	4.5	C	24	3	27
F	Eclipse	2000	2250	15	15	5	E	17	2	19
CD-1	Leece-Neville	2250	2750	12–15	25	4.5	C.C.	20	4	24
D-1	Leece-Neville	2250	2750	12–15	25	4.5	C.C.	20	3	23
D	Eclipse	2250	....	15	25	5	E	20	2	22
WD-1	Eclipse	Wind-driven	....	15	25	5	E	18[c]	3.75	21.75
CB-1[a]	Leece-Neville	2000	2250	12–15	25	5	C	28	4	32
B-1[a]	Leece-Neville	2000	2250	12–15	25	5	C	28	3	31
B	Eclipse	2000	....	15	25	5.5	E	23	2	25
E	Eclipse	2250	....	15	50	5.5	E	32	2.5	34.5
E-3	Leece-Neville	2250	2750	12–15	50	5.5	C.C.	35	3	38
E-4	Leece-Neville	2250	2750	12–15	50	5.5	C	35	3	38
C-1	Eclipse	2000	....	15	50	6	E	38	2.5	40.5
C-1[a]	Leece-Neville	2000	2250	12–15	50	6	C	44	3	47

[a] Has tachometer drive at the commutator end.

[b] Direction of rotation: C = clockwise, C.C. = counter clockwise, E = either.

[c] Includes propeller.

[d] The weights of control boxes of the Eclipse generators are for those furnished with Army orders. On Commercial and Naval orders a control box weighing 3.75 lb. is supplied.

TABLE 15 : 14

DATA ON AIRPLANE STORAGE BATTERIES

Make	Type of Battery	Dimensions in Inches			Weight Wet, Lb.	Volts	Capacity		Ampere Hour Capacity		
		Length	Width	Height			Amperes for 20 Min.*	Hours at 5 Amps.	5 Hour* Rate	1 Hour Rate	Charging Rate Finish
National....	67	4 1/4	5 7/16	11 3/8	15	6	27	3.5	...	...	1
National...	613	5 7/8	5 7/16	11 3/8	21	6	54	7.0	...	...	2
National....	127	7 1/4	5 7/16	11 3/8	27	12	27	3.5	...	...	1
Exide.......	6 TX 9-1	7 3/8	7 1/4	10 13/16	38	12	50	6.1	29.5	22	3
National....	1213	10 3/4	5 7/16	11 3/8	38	12	54	7.0	...	...	2
Exide.......	6 TX 13-1	10	7 1/4	10 13/16	53	12	75	9.7	43	31	4 1/4
National....	1225	15 1/16	5 5/8	11 3/8	72	12	108	13.6	...	...	4
Exide......	6 TX 19-1	14 1/8	7 1/4	10 13/16	70	12	113	15	65	47	6 1/2
Exide.......	6 TS 7-1	7 11/16	5 3/16	10 13/16	26	12	30	3.5	19	13	1 3/4
Exide.......	6 TS 13-1	10 5/16	5 3/16	10 13/16	36	12	60	8.1	38	26	3 1/2
Exide.......	6 TSN 25-3	15	6 9/16	11 3/8	67	12	120	17.9	75	53	7
Exide.......	4 AC 7-1	5 7/16	3 7/8	6 13/16	11	8	15	1.5	10	7	1
Exide.......	6 AC 7-1	8 1/16	3 7/8	6 13/16	17	12	15	1.5	10	7	1

* S. A. E. Ratings.

The above data were furnished by the Electric Storage Battery Co., Philadelphia, Pa., makers of Exide Batteries and the National Battery Co., St. Paul, Minn.

Miscellaneous Equipment — Data on miscellaneous items of equipment were obtained from the Air Corps, U. S. Army; Consolidated Instrument Company of America, Brooklyn, N. Y.; the Fyr-Fyter Co., Dayton, Ohio; the Johnson Airplane and Supply Co., Dayton, Ohio; the Phister Manufacturing Co., Cincinnati, Ohio; and the Pioneer Instrument Co., Brooklyn, N. Y. These data are listed in Table 15 : 15. In the column headed "Source," the source of the information is given rather than the manufacturer as the Johnson Company acts as distributor only for a number of the items listed, and the Air Corps is not a manufacturer.

TABLE 15 : 15

DATA ON MISCELLANEOUS EQUIPMENT

Item	Model	Weight	Source	Remarks
Booster Magneto	....	7.5	Johnson	Made by Bosch
Cover, Cockpit	....	4.0	Air Corps	....
Cover, Engine	....	3.0	Air Corps	....
Cover, Propeller	....	2.0	Air Corps	....
Fire Extinguisher	....	6.5	Air Corps	With bracket
Fire Extinguisher	1 Qt. Super	7.0	Fyr-Fyter	With bracket
Fire Extinguisher	....	9.75	Phister	With 1 qt. CCl_4
First Aid Kit	Travelkit	0.35	Johnson	$5\frac{3}{8}\times3\frac{3}{8}\times1\frac{3}{8}$ in.
First Aid Kit	Aerokit	1.5	Johnson	$7\frac{1}{4}\times5\frac{1}{8}\times1\frac{5}{8}$ in.
Flares, Landing	....	19.0	Johnson	Wiley Mark III
Lights, Landing	....	20.0	Consolidated	Set of 2
Lights, Landing	....	19.5	Johnson	Set of 2
Lights, Landing	....	17.0	Pioneer	Set of 2
Lights, Landing	A-2	9.5	Air Corps	36-in. streamline
Lights, Landing	A-3	8.25	Air Corps	24-in. streamline
Lights, Running		1.0	Consolidated	Set of 3
Lights, Running	Regulation	0.75	Johnson	Set of 3
Lights, Running	....	0.7	Pioneer	Set of 3
Lights, Running	New Design	1.3	Johnson	Set of 3
Lights, Running	....	1.5	Air Corps	Set of 3
Map Case	....	1.7	Air Corps	....
Oxygen Apparatus	....	20.0	Air Corps	500 liters oxygen
Safety Belt	....	0.75	Johnson	one person
Safety Belt	Double Leather	1.63	Johnson	two persons
Safety Belt	Air Corps Type	2.0	Johnson	one person
Switch, Landing Light	....	0.75	Consolidated	....
Switch, Running Light	....	0.5	Consolidated	....
Very Pistol	....	2.5	Johnson	....

15 : 9. Weights and Strengths of Materials — In the making of weight estimates and computations it is generally necessary to know the unit weights of the various materials used in airplane construction. In the case of materials used in the primary structure it is also necessary to know the unit stresses of various kinds that may be used for design purposes. Although these unit stresses cannot be classified logically as

weight data, some of them for which no logical place was found in the preceding chapters are given for convenience in the following tables.

Table 15 : 16 gives the unit weights of a number of miscellaneous materials often met with in airplane construction. These data were compiled from Wright Field Serial Report 2868, "Variation in Weights of Government Furnished Equipment" by H. L. Pfau, the "Pocket Companion" of the Carnegie Steel Co., data sheets of the Curtiss Aeroplane and Motors Co., the catalog of the Johnson Airplane and Supply Co. and the notebook of Mr. A. S. Mullgardt. For each item it was attempted to give the unit weight that would be most convenient for practical use.

TABLE 15 : 16

WEIGHTS OF MISCELLANEOUS MATERIALS, IN POUNDS

Material	Per Cu. ft.	Per Cu. in.	Per Gallon
Air	0.08071	.....	.....
Alcohol	49	0.0284	6.6
Aluminum, cast	162	0.094	.....
Asbestos	154	0.089	.....
Balsa	9.5	0.0055	.....
Benzol	52	0.0303	7.0–7.2
Brass, cast	505	0.292	.....
Brass, rolled	525	0.304	.....
Brass, sheet	527	0.305	.....
Bronze	509	0.295	.....
Celluloid	87	0.0506	.....
Copper	556	0.322	.....
Felt	5.2	0.003	.....
Fiber, hard[1]	80	0.0463	.....
Fuel, mixed[2]	49	0.0282	6.5
Gasoline	45	0.0260	5.9–6.0
Glass, common	156	0.0903	.....
Glass, plate	161	0.0932	.....
Gun metal	553	0.320	.....
Lead	710	0.410	.....
Leather, dry	53.6	0.031	.....
Leather, greased	63.9	0.037	.....
Magnalite	177	0.1025	.....
Mercury	849	0.491	.....
Monel metal	556	0.322	.....
Nickel	565	0.327	.....
Oil	56	0.325	7.5
Paper	58	0.0336	.....
Rubber[3]	70	0.0405	.....
Rubber shock disks	71	0.041	.....
Tin	459	0.266	.....
Water, fresh	62.4	0.0361	8.3
Water, sea	64	0.0370	8.5
White metal[4]	456	0.264	.....
Zinc	440	0.255	.....

[1] Bakelite and vulcanite.
[2] 50 per cent benzol, 50 per cent gasoline.
[3] Average as used in airplanes.
[4] Babbit.

TABLE 15 : 16—*Continued*

WEIGHTS OF MISCELLANEOUS MATERIALS, IN POUNDS

Part 2

Material	Sq. yd.	Sq. ft.	Sq. in.
Asbestos, 1/8 in. sheet	.....	0.76	0.0053
Brazing, dip	.....	0.144	0.0010
Celluloid, 0.060 in. sheet	.....	0.458	0.0032
Celluloid, 1/16 in. sheet	.....	0.47	0.0033
Celluloid, 0.100 in. sheet	.....	0.761	0.0053
Celluloid, 1/8 in. sheet	.....	0.95	0.0066
Dope	0.266	0.0295	0.000205
Enamel, aluminum, 1 coat	.....	0.0035	0.000024
Fabric, Batiste	0.162	0.018	0.000125
Fabric, Cotton Grade A	0.250	0.0280	0.000194
Fabric, Special Airplane	0.218	0.024	0.000167
Felt, 3/8 in.	.....	0.216	0.0015
Felt, 1/2 in.	.....	0.288	0.0020
Glass, Triplex, 3/16 in.	.....	1.97	0.0137
Glass, Triplex, 7/32 in.	.....	2.66	0.0185
Glass, Triplex, 5/16 in.	.....	3.20	0.0222
Lacquer, 2 coats on aluminum	.....	0.006	.....
Leather, imitation	.....	0.15	0.00104
Paint, Bitumastic aluminum, 1 coat	.....	0.0035	.....
Paint, Khaki on cowling	.....	0.01	.....
Screen, brass, 20 mesh	.....	0.375	0.0026
Screen, brass, 50 mesh	.....	0.316	0.0022

Table 15 : 17 gives the weights and allowable unit stresses for various species of wood used in airplane construction. Table 15 : 18 gives the data needed to compute the weight of plywood. Both of these tables are taken from "Airplane Design" published by the Air Corps, but were originally prepared by the Forest Products Laboratory.

Table 15 : 19 gives the weights and allowable unit stresses for aluminum alloy and various grades of steel used in aircraft. The unit stresses in this table were taken from the 1927 edition of the Air Corps Handbook of Instructions for Airplane Designers.

TABLE 15 : 17

Strength Values of Various Woods for Use in Airplane Design
(Based on 15 per cent moisture content)

Common and Botanical Names	Specific Gravity Based on Volume and Weight when Oven-Dry: Average	Specific Gravity: Minimum Permitted	Weight at 15% Moisture, Lb. per cu. ft.	Static Bending: Fiber Stress at Elastic Limit,[1] Lb. per sq. in.	Static Bending: Modulus of Rupture,[1] Lb. per sq. in.	Static Bending: Modulus of Elasticity,[2] 1000 Lb. per sq. in.	Static Bending: Work to Max. Load, In.-lb. per cu. in.	Compression Parallel to Grain: Fiber Stress at Elastic Limit,[1,3] Lb. per sq. in.	Compression Parallel to Grain: Max. Crushing Strength,[1] Lb. per sq. in.	Compression Perpendicular to Grain,[4] Lb. per sq. in.	Shearing Strength Parallel to Grain,[5] Lb. per sq. in.	Hardness Side Load Req'd to Imbed 0.444-In. Ball to Half Its Diameter, Lb.
Hardwoods												
Ash, black (*Fraxinus nigra*)	0 53	0 48	35	6,400	11,900	1340	14.3	4050	5400	1260	1050	760
Ash, commercial white (*Fraxinus sp.*)[6]	0.62	0 56	41	8,900	14,800	1460	14.2	5250	7000	2250	1380	1180
Basswood (*Tilia Americana*)	0 40	0 36	26	5,600	8,600	1250	6.6	3370	4500	620	720	370
Beech (*Fagus atropunicea*)	0 66	0 60	44	8,200	14,200	1440	13.5	4800	6500	1670	1300	1060
Birch (*Betula sp.*)[7]	0 68	0.58	44	9,500	15,500	1780	18 2	5480	7300	1590	1300	1100
Cherry, black (*Prunus serotina*)	0.53	0 48	36	8,500	12,500	1330	11.7	5100	6800	1170	1180	900
Cottonwood (*Populus deltoides*)	0 43	0 39	29	5,600	8,600	1190	7.4	3520	4700	650	660	410
Elm, cork (*Ulmus racemosa*)	0.66	0.60	45	7,900	15,000	1340	19.3	5180	6900	2090	1360	1230
Gum, red (*Liquidambar styraciflua*)	0.53	0 48	34	7,500	11,600	1290	10 9	4050	5400	1190	1100	650
Hickory (true hickories (*Hicoria sp.*)[8]	0 79	0.71	51	10,600	19,300	1860	27 5	6520	8700	3100	1440	....
Mahogany, African (*Khaya sp.*)	0.47	0.42	32	7,900	10,800	1280	8 0	4280	5700	1400	980	720
Mahogany, true (*Swietenia sp.*)[9]	0.51	0 46	34	8,800	11,600	1260	7.3	4880	6500	1760	860	790
Maple, sugar (*Acer saccharum*)	0 67	0.60	44	9,500	15,000	1600	13 7	5620	7500	2170	1520	1270
Oak, commercial white and red (*Quercus sp.*)[10]	0.69	0 62	45	7,800	13,800	1490	13 6	4950	6600	1870	1300	1240
Poplar, yellow (*Liriodendron tulipifera*)	0 43	0 38	28	6,000	9,100	1300	6 5	3750	5000	810	800	420
Walnut, black (*Juglans nigra*)	0.56	0 52	39	10,200	15,100	1490	11 4	5700	7600	1730	1000	990
Conifers												
Cedar, incense (*Libocedrus decurrens*)	0 36	0 32	25	6,000	8,700	1020	5.6	4320	5400	900	650	450
Cedar, Port Orford (*Chamaecyparis lawsoniana*)	0 44	0 40	30	7,400	11,000	1520	8.7	4880	6100	1030	760	520
Cedar, western red (*Thuja plicata*)	0.34	0 31	23	5,100	7,800	1030	5 8	4000	5000	800	630	320
Cedar, white (northern) (*Thuja occidentalis*)	0 32	0 29	22	4,700	6,600	700	4.9	3040	3800	560	610	300
Cypress, bald (*Taxodium distichum*)	0 48	0.43	32	7,100	10,500	1270	7 7	4960	6200	1230	720	480
Douglas fir (*Pseudotsuga taxifolia*)	0 51	0.45	34	8,000	11,500	1700	8 1	5600	7000	1300	810	620
Pine, Norway (*Pinus resinosa*)	0.51	0.46	34	8,500	11,900	1560	8.9	5280	6600	1080	870	520
Pine, sugar (*Pinus lambertiana*)	0.38	0.34	26	5,600	8,000	1040	5.4	3680	4600	810	730	370
Pine, western white (*Pinus monticola*)	0 42	0.38	27	6,000	9,300	1310	7.9	4240	5300	750	640	360
Pine, white (*Pinus strobus*)	0 38	0 34	26	5,900	8,700	1140	6.3	3840	4800	780	640	380
Spruce (*Picea sp.*)[11]	0 40	0.36	27	6,200	9,400	1300	7.8	4000	5000	840	750	440

From N A.C.A. Technical Note #296 " Bearing Strength of Wood Under Steel Aircraft Bolts " by G. W. Trayer of the ForestProducts Laboratory.
For Notes see next page.

Notes for Table 15:17

[1] The average values for fiber stress at elastic limit and modulus of rupture in static bending, and fiber stress at elastic limit and maximum crushing strength in compression parallel to grain have been multiplied by two factors to obtain values for use in design. A statement of these factors and of the reasons for their use follow. It was thought best, in fixing upon strength values for use in design, to give some influence to the variability of wood and to the fact that a greater number of values are below the average than above it, and the most probable value (as represented by the mode of the frequency curve) was accordingly decided upon as the basis for design figures. From a study of the ratios of most probable to average values for three species (Sitka spruce, Douglas fir, and white ash), 0.94 was adopted as the best value of this ratio for general application to the properties in question.

The stress that wooden members can carry depends on its duration. A factor of 1.17 has been applied to test results to get values of the stress which can be sustained for a period of three seconds, it being assumed that the maximum load will not be maintained for a longer period.

[2] The values given are the most probable values (92 per cent of the average) of the apparent modulus of elasticity (E_c) as obtained by substituting results from tests of 2 by 2-in. beams on a 28-in. span with load at the center in the formula $E_c = Pl^3/48\triangle I$. The use of these values of E_c in the usual formulas will give the deflection of beams of ordinary length with but small error. For exactness in the computation of deflections of I and box beams, particularly for short spans, the formula which takes into account shear deformations (see National Advisory Committee for Aeronautics Report No. 180, "Deflection of Beams with Special Reference to Shear Deformations") should be used. This formula involves E_t the true modulus of elasticity in bending, and F, the modulus of rigidity in shear. Values of E_t may be obtained by adding 10 per cent to the values of E_c as given in the table. If the I or box beam has the grain of the web parallel to the axis of the beam or parallel and perpendicular thereto, as in some plywood webs, the value of F may be taken as $E_t/16$ or $E_c/14.5$. If the web is of plywood with the grain at 45 degrees to the axis of the beam F may be taken as $E_t/5$ or $E_c/4.5$.

[3] Design values for fiber stress at elastic limit in compression parallel to grain were obtained by multiplying the values of maximum crushing strength as given in the next column by factors as follows: 0.75 for hardwoods — 0.80 for conifers. Values as given are to nearest 10 pounds.

[4] Wood does not exhibit a definite ultimate strength in compression perpendicular to grain, particularly when the load is applied over only a part of the surface, as at fittings. Beyond the elastic limit the load continues to increase slowly until the deformation and crushing become so severe as to seriously damage the wood in other properties. Figures in this column were obtained by applying a duration of stress factor of 1.17 (see Note 1) to the average elastic limit stress and then adding 33⅓ per cent to get design values comparable to those for bending, compression parallel to grain, and shear as listed in the table.

[5] Values in this column are for use in computing resistance of beams to longitudinal shear. They are obtained by multiplying average values by 0.75. This factor is used because of the variability in strength and in order that failure by shear may be less probable than failure from other causes. Furthermore, tests have shown that because of the favorable influence upon the distribution of stresses resulting from limiting shearing deformations, the maximum strength weight ratio and minimum variability in strength are attained when I and box beams are so proportioned that the ultimate shearing strength is not developed and failure by shear does not occur.

[6] Includes white ash (*F. Americana*), green ash (*F. lanceolata*), and blue ash (*F. quadrangulata*).

[7] Includes sweet birch (*B. lenta*) and yellow birch (*B. lutea*).

[8] Includes big shellbark hickory (*H. laciniosa*), mockernut hickory (*H. alba*), pignut hickory (*H. glabra*), and shagbark hickory (*H. ovata*).

[9] Includes material from Central America and Cuba.

[10] Includes white oak (*Q. alba*), bur oak (*Q. macrocarpa*), cow oak (*Q. michauxii*), post oak (*Q. minor*), red oak (*Q. rubra*), Spanish (highland) oak (*Q. digitata*), laurel oak (*Q. laurifolia*), water oak (*Q. nigra*), Spanish (lowland) oak (*Q. pagodaefolia*), willow oak (*Q. phellos*), yellow oak (*Q. velutina*).

[11] Includes red spruce (*P. rubens*), white spruce (*P. canadensis*), and Sitka spruce (*P. sitchensis*).

TABLE 15 : 18

Weights of Veneer

(In ounces per square foot for single ply veneer. Thickness in inches)

Species	Sp. Grav.	Per Cent Moisture	1/64	1/55	1/48	1/32	1/28	1/24	1/16	1/12	1/10	1/8	1/6	3/16	1/4
Basswood	0.38	8.4	0.49	0.58	0.68	0.99	1.13	1.32	1.98	2.64	3.16	3.96	5.28	5.94	7.92
Birch, Yellow	0.63	9.6	0.82	0.95	1.09	1.64	1.87	2.19	3.28	4.37	5.24	6.56	8.74	9.84	13.12
Cottonwood	0.43	4.7	0.56	0.65	0.75	1.12	1.28	1.49	2.24	2.98	3.58	4.47	5.97	6.71	8.96
Gum, Red	0.49	11.3	0.64	0.74	0.85	1.28	1.46	1.70	2.55	3.40	4.08	5.10	6.80	7.60	10.20
Maple, Silver	0.48	8.2	0.62	0.73	0.83	1.25	1.43	1.67	2.50	3.33	4.00	5.00	6.66	7.50	10.00
Maple, Sugar	0.62	10.5	0.81	0.94	1.08	1.61	1.85	2.15	3.23	4.30	5.16	6.46	8.60	9.69	12.91
Poplar, Yellow	0.41	6.1	0.53	0.62	0.71	1.07	1.22	1.42	2.13	2.84	3.41	4.27	5.69	6.40	8.54
Fir, Douglas	0.44	9.4	0.57	0.67	0.76	1.15	1.31	1.53	2.29	3.05	3.66	4.58	6.10	6.87	9.16
Spruce, Sitka	0.38	8.9	0.49	0.58	0.66	0.99	1.13	1.32	1.98	2.64	3.16	3.96	5.28	5.94	7.92
Mahogany, African	0.46	5.6	0.60	0.70	0.80	1.20	1.37	1.60	2.39	3.19	3.83	4.79	6.38	7.18	9.58
Mahogany, True	0.49	7.9	0.65	0.75	0.85	1.28	1.46	1.70	2.55	3.50	4.08	5.10	6.80	7.66	10.20

Weight of glue per square foot: Blood albumin about 0.3 oz. Casein about 0.4 oz.

Example: To get the weight of a square foot of 5-plywood consisting of 1 ply of 1/12 in. basswood, 2 plies of 1/16 in. basswood, and 2 plies of 1/24 in. yellow birch for faces, at 12 per cent moisture, glued with casein glue.

Weight = $(1 \times 2.64 + 2 \times 1.98 + 2 \times 2.19)\ 1.12 + 4 \times 0.4 = 13.90$ oz.

The weight of wood is quite variable, so that while the table given represents the average weights of material tested at the laboratory, large variations from these figures may be expected in individual pieces of veneer.

The example above is slightly in error through neglecting the change in volume between the moisture content at 12 per cent and the moistures listed in the table.

Data supplied by Forest Products Laboratory.

TABLE 15 : 19

Material	Weight in Pounds			Properties in Thousands of Pounds per Square Inch					
	Cu. ft.	Cu. in.	1 in.×1 in.× 12 in.	Ultimate Tension[f]	Yield Point	Bearing[g]	Shear	Block Comp.[h]	Mod. of Elasticity
Aluminum Alloy-Sheet	175	0.1011	1.213	55	30	75[i]	27[j]	40	10,000
Aluminum Alloy-Bar	175	0.1011	1.213	50[k]	25[k]	75	30	40	10,000
Aluminum Alloy-Rivet	175	0.1011	1.213	30[l]	..	..	30[m]	..	..
Aluminum Alloy-Tubing	175	0.1011	1.213	55	30	75	27	40	10,400
Steels:									
S.A.E. 1025 sheet, tube, and bar	490	0.2833	3.40	55	36	90	35	55	28,000
Alloy[a], sheet and bar, not heat-treated[b]	490	0.2833	3.40	65	45	110	40	65	29,000
C.M. sheet and tubing[c]	490	0.2833	3.40	95	60	140	60	95	29,000
Alloy[a], near welding when welded after heat-treatment	490	0.2833	3.40	80[n]	60	125	50	80	29,000
Alloy[a], heat-treated[d]	490	0.2833	3.40	100	80	140	65	100	29,000
Alloy[a], heat-treated[d]	490	0.2833	3.40	125	105	175	80	125	29,000
Alloy[a], heat-treated[d]	490	0.2833	3.40	150	125	190	100	150	29,00(
Alloy[a], heat-treated[d]	490	0.2833	3.40	180	140	200	115	180	29,000
Alloy[a], heat-treated[de]	490	0.2833	3.40	200	150	220	120	200	29,000

[a] The alloy steels for which the strength values above may be used are: 6130 chrome-vanadium sheet, 6135 and 4130 bar stock, and chrome molybdenum sheet and tubing conforming to Air Corps specifications. Welding and brazing may be practiced on any of these steels.

[b] For bars not larger than 1.5 in. in diameter or thickness. For bars over 1.5 in. in diameter or thickness, use the figures for 1025 steel.

[c] For use with chrome-molybdenum steel sheet or tubing conforming to the Air Corps specifications for these materials. These properties can be obtained by normalizing or by quenching and drawing.

[d] For use with heat-treated alloy steel not welded. When assemblies are made up by welding or brazing alloy steel after heat-treatment, the unit stresses to be used in locations adjacent to the welded or brazed portions should be those listed for alloy steel near welding when welded after heat-treatment for sheet and tubing, and those listed for alloy steel not heat-treated for bar. In locations that would not be affected by the heat of welding or brazing the values for heat-treated alloy steel not welded corresponding to the heat-treatment employed may be used. No general rule can be given for the distance from the weld or brazed joint to which the effect of the heat of the operation will extend, but each case must be considered on its own merits. The unit stresses in and near joints heat-treated after welding

should not exceed 80 per cent of the standard properties of the steel corresponding to the heat-treatment used, except the unit stress in the deposited metal of the welded seam which should not exceed 50,000 lb. per sq. in. The thickness of the welded seam should not be assumed as more than 75 per cent greater than the thickness of the welded stock.

[e] The thickness of tubing heat treated to 200,000 lb. per sq. in. ultimate tension should not be less than 0.120 in.

[f] The ultimate tension values may be used for the moduli of rupture in bending of closed sections like tubes. No higher values should be used for that property unless justified by specific tests. For shapes subjected to local buckling lower values should be used for the modulus of rupture, the value for each shape being determined by test.

[g] The bearing values in this table should be used only in rigid joints in which there is no possibility of relative movement between the parts joined without deformation of those parts. In general this classification covers joints made with two or more rivets, bolts, or pins. If there is possibility of relative movement between the parts joined without their deformation, the bearing stresses should be reduced one-third. In joints where the shock may be considerable, as in chassis to fuselage fittings and some of the connections in the control systems, it is advisable to reduce the bearing values of the table by one-half or more.

[h] The block compression values of this table should be used only for members with a slenderness ratio L/ρ less than 10.0.

[i] For sheets less than 0.035 in. thick, use the following formula for the allowable bearing stress:

$$F_b = \frac{75{,}000\, t}{0.035}$$

where F_b is the allowable bearing stress and t is the thickness of the plate. The bearing values given in Table 10 : 5, page 173, were computed on this basis.

[j] For thickness of 0.0625 in. or less use 20,000 lb. per sq. in.

[k] For bar stock $\frac{3}{4}$ in. or less in diameter, use 55,000 lb. ultimate tension and 36,000 lb. per sq. in. yield point.

[l] For annealed rivets use 10,000 lb. per sq. in.

[m] For solid hardened rivets only. For hollow hardened rivets use 20,000 lb. and for annealed rivets, solid or hollow, use 10,000 lb. per sq. in.

[n] In the design of welded chrome-molybdenum steel tubing fuselages, wing spar trusses, compression ribs, and similar members, the average tensile stress in each tube should not be more than 80,000 lb. per sq. in. In such structures, tubes entering joints at which more than six members converge should not be subjected to average tensile stresses greater than 60,000 lb. per sq. in. unless the joint is adequately reinforced by gussets or their equivalent. In this connection a tube that is continuous through the joint should be counted as two members.

15 : 10. Weight Control — Once the preliminary weight estimate has been completed, the designer should adopt the estimated gross weight, or a slightly larger figure to allow for contingencies, as his "bogie" weight, and make every effort to prevent the actual weight from being greater than this figure. The most satisfactory way of accomplishing this end is by "budgeting" his weights. To do this the airplane is divided into a number of units, and these units into sub-units to each of which a part of the gross weight is assigned. Normally these units and sub-units and the weights assigned to them would be those appearing in the preliminary weight estimate.

As the design progresses it will be possible to make more precise estimates of weight through computations based on the design sizes used. As soon as the design sizes of the more important members of any part have been selected, a computation of the weight of the part should be made. At this stage the computation cannot be complete, but it should show whether it is reasonable to expect that the whole part can be built with less than the weight assigned to it. If this or any later computation shows that the part will be heavier than was assumed, the designer must either redesign the part to get within the limits he has set himself, or else must find some other part that is so much lighter than its bogie that the bogies for the two parts can be modified without increasing the gross weight.

When possible it is desirable to divide the main groups of items into sub-groups. In this way any tendency of the design to run to excessive weight can be detected more quickly, and also more freedom can be allowed in changing allotments between the sub-groups of the same major group than between different main groups.

A suggested classification of the gross weight for use in the weight budget record is given in Table 15 : 20.

TABLE 15 : 20

CLASSIFICATION OF WEIGHTS

USEFUL LOAD
- Crew
- Passengers
- Fuel
- Oil
- Cargo
  - Mail
  - Baggage
  - Express
- Special Equipment
  - Heating
  - Navigation
  - Photographic
  - Radio
  - Safety (Parachutes, Flares, etc.)
  - Miscellaneous

  (Heating through Miscellaneous:) Each group itemized

WEIGHT EMPTY

STRUCTURE

Wing Group

<table>
<tr>
<td>Upper Outer Panels
Upper Center Panels
Lower Outer Panels
Lower Inner Panels
Upper Ailerons
Lower Ailerons</td>
<td>Each sub-divided, as far as pertinent, into</td>
<td>Spars — Front
Spars — Rear
Spars — Auxiliary
Leading Edge
Trailing Edge
Ribs
Drag Struts or Ribs
Drag Wires
Wing Hinges
Connecting Fittings
Bolts, Nuts, etc.
Blocks or Gusset Plates
Nose Covering
Control Horns
Wing Tip Bow
Glue, Nails, etc. (Assembly)
Built-in Supports for Electrical, Radio, and other material
Protective Coatings used on Skeleton
Miscellaneous Skeleton Items
Covering
Dope
Windows, and Frames
Walkways</td>
</tr>
</table>

Interplane Bracing

<table>
<tr>
<td>Struts
Wires
Fairing
Clips and Connections</td>
<td>These may be sub-divided into Front, Rear, Intermediate, etc.</td>
</tr>
</table>

Tail Group

<table>
<tr>
<td>Stabilizer
Elevators
Fin
Rudder</td>
<td>Each sub-divided as far as pertinent, into</td>
<td>Spar — Front
Spar — Rear
Leading Edge
Trailing Edge
Ribs
Drag Struts or Ribs
Drag Wires
Fittings
Bolts, Nuts, etc.
Control Horns
Blocks or Gusset Plates
Nose Covering
Glue, Nails, Rivets, etc. (Assembly)
Miscellaneous Skeleton Items
Covering
Dope
Paint</td>
</tr>
</table>

Bracing

- External Struts
- External Wires

Connections

- Hinge Pins
- Bolts, Nuts, etc.

Body Group

Fuselage
Engine Mount (if detachable)
Nacelles
Nacelle Supports

Each sub-divided, as far as pertinent, into

- Longerons
- Struts
- Wires
- Fittings
- Fire-wall
- Built-in Supports for Equipment
- Turtle-back, Fairing-strips, etc., for supporting cowls and fabric covering
- Steps and Grips
- Bolts, Nuts, etc.
- Walkways, External
- Engine Ring or Bearers
- Rubber Pads or Vibration Insulators
- Protective Coatings for Skeleton
- Miscellaneous Skeleton Items
- Fabric Covering, inc. inspection doors
- Metal Cowling, inc. inspection doors and engine shutters
- Hatches and Windows
- Dope and Paint

Hull
Main Floats
Wing Tip Floats

Each sub-divided, as far as pertinent, into

- Longitudinal Framing
  - Keel
  - Keelson
  - Chine
  - Struts
  - Longerons
  - Deck Stringers — Center
  - Deck Stringers — Side
  - Longitudinal Bulkhead
- Transverse Framing
  - Water-tight Bulkheads
  - Non-water-tight Bulkheads
  - Floors
  - Deck Frames
  - Struts
- Fittings
- Bolts, Nuts, Rivets, etc.
- Nose Bumper
- Stern Post
- Rubbing Strip
- Step Block
- Spray Strip
- Hand Hole Frames and Covers
- Walkways, External
- Hatches and Windows
- Built-in Supports for Equipment
- Bottom Plating
- Deck Plating
- Protective Coatings
- Water-stop Material
- External Struts — Floats
- External Wires — Floats

Mooring, Towing, Handling Gear
- Mooring and Towing Fittings
- Anchor
- Anchor Lines
- Mooring and Towing Pendant
- Gear for Retrieving and Stowing Anchor
- Wing Handling Lines

Landing Gear Group

Chassis
- Main Struts
- Wires
- Axle
- Wheels, inc. Brakes
- Shock Absorber
  - Shock-absorber Cord
  - Rubber Discs
  - Oleo Leg
  - Oil
- Tires
- Wheel Retracting Mechanism
- Fittings
- Mud Guards
- Fairings
- Bolts, Nuts, etc.
- Protective Coatings

Tail Skid Assembly
- Tail skid
- Tail skid shoe
- Tail Wheel
- Tail Skid Shock Absorber
- Controls For Steering

Wing Skids

POWER PLANT

Engines

Engine, dry

Engine Accessories

- Air Intake Pipes
- Air Heating Devices
- Exhaust Manifold

Engine Controls

- Sector and Levers
- Bell cranks
- Bearings and Supports
- Rods
- Cables
- Pulleys
- Radiator Controls
- Engine-cowl Shutter Controls

Propeller

- Blades
- Hub
- Spinner
- Adjustable-pitch Mechanism and Controls

Starting System

- Starter
- Brackets for Crank
- Starter Crank
- Primer
- Primer Piping and Fittings

Cooling System
- Radiator
- Shutters
- Expansion Tank
- Water in
  - Motor
  - Radiator
  - Expansion Tank
  - Piping
  - Oil Cooler or Regulator

Fuel System, *Oil System* — Each sub-divided, as far as pertinent, into:
- Tanks — Main
- Tanks — Gravity
- Collector Tank or Manifold
- Dump Valve
- Dump Valve controls
- Tank Release.
- Tank Release Controls
- Oil Cooler or Regulator
- Fuel Pumps
- Valves
- Strainers
- Piping and Fittings
- Supports

Fixed Equipment

*Instruments**
- Air Speed Indicator
- Air Speed Pitot tube
- Airspeed Tubing
- Inclinometers
- Turn Indicator
- Turn Indicator Venturi Tube
- Turn Indicator Tubing
- Altimeter
- Compass, Magnetic
- Clock and Mount
- Earth Inductor Compass
- Water Thermometer and Tubing
- Oil Thermometer and Tubing
- Oil Pressure Gauge and Tubing
- Fuel Pressure Gauge and Tubing
- Fuel Quantity Gauge and Operating Mechanism
- Tachometer
- Tachometer Shafting or Generator and Wiring

Surface Controls
- Control Stick or Column (Pilot)
- Control Stick or Column (Relief pilot)
- Control Stick or Column Supports
- Members connecting Control Sticks or Columns
- Rudder Bar or Pedals (Pilot)
- Rudder Bar or Pedals (Relief pilot)
- Rudder Bar Supports
- Aileron Controls, Elevator Controls, Rudder Controls — Each sub-divided into:
  - Cables
  - Rods
  - Pulleys
  - Bell Cranks
  - Fairleads or Guides
  - Supports (Removable)
  - Struts or Wires used to interconnect control surfaces
  - Bolts, Nuts, etc.

* Some of these instruments may be classified under Special Equipment in any specific case.

- Stabilizer Adjusting Mechanism
  - Special Stabilizer Members
  - Cables and Rods
  - Wheel
  - Pulleys, Bolts, Nuts, etc.

Furnishings

- Seats and Adjusting Mechanisms
  - Pilots
  - Passengers
- Flooring
  - Pilot's Cockpit
  - Cabin, or Passenger's Cockpit
- Windshields
- Instrument Boards
- Tool-kit Locker
- Radio Locker
- Miscellaneous Lockers
- Shelves
- Baggage Compartment
- Mail Compartment
- Hoisting Sling
- Miscellaneous Living Accomomodations

Electrical Equipment

- Generator
- Generator Control box
- Battery
- Battery Container and Supports
- Ammeter
- Voltmeter
- Navigation Lighting
  - Wiring and Conduits
  - Bulb, Base, and Reflector
  - Switches
- Landing Lights
  - Wiring and Conduits
  - Bulbs
  - Lights
  - Supports
  - Retracting Mechanism
  - Switches
- Interior Lighting
  - Wiring and Conduits
  - Bulbs, Bases, and Reflectors
  - Switches
- Panels
- Circuit Breakers
- Booster Magneto, Wiring, and Control for starting engine
- Ignition Switch
- Ignition Master Switch
- Ignition Switch Rods, Wiring and Conduit
- Wiring, Conduit, and Connection Panel for wing tip flares

If the preliminary weight computation based on the partial design indicates a satisfactory weight, the detailed design should be completed and a new estimate made on the basis of the detailed drawings. In making this computation the designer should include every source of weight such as rivet heads, paint, nails, glue, nuts, and clevises, and an allowance of from 2 to 4 per cent for omissions, effects of manufacturing

tolerances, lack of precision of computations, and deviations of the unit weights of the materials actually used from the standard values. It is good practice to list the weight of each item on the drawing of the item in order that the weight of the assembly can be computed easily. Often this weight estimate will be very tedious to make and is omitted. Such omission is not likely to be serious if the bogie value was selected with judgment and the design has been carefully made, or if the estimate based on the major dimensions of the main members was conservative.

Whether the computation outlined in the preceding paragraph is made or not, as soon as each item is constructed it should be weighed. If the actual weight is less than the bogie, a margin is provided that can be used to increase the bogie weights of other parts. If the actual weight is the greater and sufficient margin has not been accumulated from light items to make up the difference, either the part must be scrapped and resdeigned or the remainder of the airplane designed with such care that the needed margins will be obtained.

For ease in visualizing the weight situation at any moment it is advisable to post the weight estimates on a tabular form with one line for each item and a column for each estimate. Several columns should be provided for the bogie weights, as they will normally be revised from time to time when certain items prove over or under weight. The last column at the right would naturally be used for the actual weight.

The mechanical operation of this system of budgeting weights is not at all difficult and the principles underlying it are very easy to understand. The only hard thing about it is developing the strength of character required to stick to it, and scrap parts that prove overweight.

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